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CHAPTER III

Derivation of the equations of motion of a compressible viscous fluid

(Navier-Stokes equations)†

a. Fundamental equations of motion and continuity applied to fluid flow

We shall now proceed to derive the equations of motion of a compressible, viscous, Newtonian fluid. In the general case of three-dimensional motion, the flow field is specified by the velocity vector

$$\boldsymbol{w} = \boldsymbol{i}\, u + \boldsymbol{j}\, v + \boldsymbol{k}\, w$$

where u, v, w are the three orthogonal components, by the pressure p, and by the density ϱ, all conceived as functions of the coordinates x, y, z, and time t. For the determination of these five quantities there exist five equations: the continuity equation (conservation of mass), the three equations of motion (conservation of momentum) and the thermodynamic equation of state $p = f(\varrho)$.‡

The equation of continuity expresses the fact that for a unit volume there is a balance between the masses entering and leaving per unit time, and the change in density. In the case of non-steady flow of a compressible fluid this condition leads to the equation:

$$\frac{\mathrm{D}\varrho}{\mathrm{D}t} + \varrho \operatorname{div} \boldsymbol{w} = \frac{\partial \varrho}{\partial t} + \operatorname{div}(\varrho\, \boldsymbol{w}) = 0\,, \qquad (3.1)$$

whereas for an incompressible fluid, with $\varrho = \text{const}$, the equation of continuity assumes the simplified form

$$\operatorname{div} \boldsymbol{w} = 0\,. \qquad (3.1\,\mathrm{a})$$

The symbol $\mathrm{D}\varrho/\mathrm{D}t$ denotes here the substantive derivative which consists of the local contribution (in non steady flow) $\partial\varrho/\partial t$, and the convective contribution (due to translation), $\boldsymbol{w}\cdot\operatorname{grad}\varrho$.

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† In the Sixth Edition this chapter has been revised by the Translator at the Author's invitation.

‡ If the equation of state contains temperature as an additional variable, a further equation is supplied by the principle of the conservation of energy in the form of the First Law of Thermodynamics; *cf.* Chap. XII.

The equations of motion are derived from Newton's Second Law, which states that the product of mass and acceleration is equal to the sum of the external forces acting on the body. In fluid motion it is necessary to consider the following two classes of forces: forces acting throughout the mass of the body (gravitational forces) and forces acting on the boundary (pressure and friction). If $\boldsymbol{F} = \varrho\, \boldsymbol{g}$ denotes the gravitational force per unit volume ($\boldsymbol{g}$ = vector of acceleration due to gravity) and $\boldsymbol{P}$ denotes the force on the boundary per unit volume, then the equations of motion can be written in the following vector form

$$\varrho \frac{\mathrm{D}\boldsymbol{w}}{\mathrm{D}t} = \boldsymbol{F} + \boldsymbol{P} \tag{3.2}$$

with

$$\boldsymbol{F} = \boldsymbol{i}\, X + \boldsymbol{j}\, Y + \boldsymbol{k}\, Z \qquad \text{body force} \tag{3.3}$$

and

$$\boldsymbol{P} = \boldsymbol{i}\, P_x + \boldsymbol{j}\, P_y + \boldsymbol{k}\, P_z \qquad \text{surface force}\,. \tag{3.4}$$

The symbol $\mathrm{D}\boldsymbol{w}/\mathrm{D}t$ denotes here the substantive acceleration which, like the substantive derivative of density, consists of the local contribution (in non-steady flow) $\partial \boldsymbol{w}/\partial t$, and the convective contribution (due to translation) $\mathrm{d}\boldsymbol{w}/\mathrm{d}t = (\boldsymbol{w}\cdot\mathrm{grad})\,\boldsymbol{w}$†

$$\frac{\mathrm{D}\boldsymbol{w}}{\mathrm{D}t} = \frac{\partial \boldsymbol{w}}{\partial t} + \frac{\mathrm{d}\boldsymbol{w}}{\mathrm{d}t}\,.$$

The body forces are to be regarded as given external forces, but the surface forces depend on the *rate* at which the fluid is *strained* by the velocity field present in it. The system of forces determines a *state of stress*, and it is now our task to indicate the relationship between stress and rate of strain, noting that it can only be given empirically. In our present derivation we shall restrict attention to *isotropic, Newtonian fluids* for which it may be assumed that this relation is a linear one. All gases and many liquids of interest in boundary-layer theory, in particular water, belong to this class. A fluid is said to be isotropic when the relation between the components of stress and those of the rate of strain is the same in all directions; it is said to be Newtonian when this relation is linear, that is when the fluid obeys Stokes's law of friction. In the case of isotropic, elastic solid bodies, experiment teaches that the state of stress depends on the magnitude of strain itself, most engineering materials obeying Hooke's linear law which is somewhat analogous to Stokes's law. Whereas the relation between stress and strain for an isotropic elastic solid involves two constants which characterize the properties of a given material (e. g. elastic modulus and Poisson's ratio), the relation between stress and rate of strain in an isotropic fluid involves a single constant (the viscosity, μ) as long as relaxation phenomena do not occur within it, as we shall see in Sec. III e.

† In order to express the vector $(\boldsymbol{w}\cdot\mathrm{grad})\,\boldsymbol{w}$ in an arbitrary system of coordinates, the following general relation should be used

$$(\boldsymbol{w}\cdot\mathrm{grad})\,\boldsymbol{w} = \mathrm{grad}\,\tfrac{1}{2}\,\boldsymbol{w}^2 - \boldsymbol{w} \times \mathrm{curl}\,\boldsymbol{w}\,,$$

where $\boldsymbol{w}^2 = \boldsymbol{w}\cdot\boldsymbol{w}$.

b. General stress system in a deformable body

In order to write down expressions for the surface forces acting on the boundary, let us imagine a small parallelepiped of volume $\mathrm{d}V = \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$ isolated instantaneously from the body of the fluid, Fig. 3.1, and let its lower left-hand vertex coincide with the point x, y, z. On the two faces of area $\mathrm{d}y \cdot \mathrm{d}z$ which are perpendicular to the x-axis there act two resultant stresses (vectors = surface force per unit area):

$$\boldsymbol{p}_x \quad \text{and} \quad \boldsymbol{p}_x + \frac{\partial \boldsymbol{p}_x}{\partial x}\,\mathrm{d}x \quad \text{respectively}\,. \tag{3.5}$$

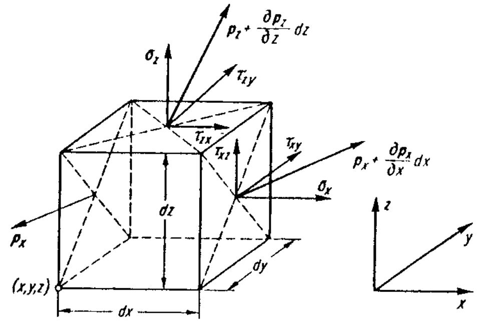

Fig. 3.1. Derivation of the expressions for the stress tensor of an inhomogeneous stress system and of its symmetry in the absence of a volumetric distribution of local moments

(Subscript x denotes that the stress vector acts on an elementary plane which is perpendicular to the x-direction.) Similar terms are obtained for the faces $\mathrm{d}x \cdot \mathrm{d}z$ and $\mathrm{d}x \cdot \mathrm{d}y$ which are perpendicular to the y- and z-axes respectively. Hence the three net components of the surface force are:

$$\text{plane} \perp \text{direction } x: \quad \frac{\partial \boldsymbol{p}_x}{\partial x} \cdot \mathrm{d}x \cdot \mathrm{d}y \cdot \mathrm{d}z$$

$$\text{,,} \quad \text{,,} \quad \text{,,} \quad y: \quad \frac{\partial \boldsymbol{p}_y}{\partial y} \cdot \mathrm{d}x \cdot \mathrm{d}y \cdot \mathrm{d}z$$

$$\text{,,} \quad \text{,,} \quad \text{,,} \quad z: \quad \frac{\partial \boldsymbol{p}_z}{\partial z} \cdot \mathrm{d}x \cdot \mathrm{d}y \cdot \mathrm{d}z\,.$$

and the resultant surface force $\boldsymbol{P}$ per unit volume is, therefore, given by

$$\boldsymbol{P} = \frac{\partial \boldsymbol{p}_x}{\partial x} + \frac{\partial \boldsymbol{p}_y}{\partial y} + \frac{\partial \boldsymbol{p}_z}{\partial z}\,. \tag{3.6}$$

The quantities $\boldsymbol{p}_x$, $\boldsymbol{p}_y$, $\boldsymbol{p}_z$ are vectors which can be resolved into components perpendicular to each face, i. e., into normal stresses denoted by σ with a suitable subscript indicating the direction, and into components parallel to each face, i. e. into shearing stresses denoted by τ. The symbol for a shearing stress will be provided with two subscripts: the first subscript indicates the axis to which the face is perpendicular, and the second indicates the direction to which the shearing stress is parallel. With this notation we have

$$\left.\begin{aligned} p_x &= i\,\sigma_x + j\,\tau_{xy} + k\,\tau_{xz} \\ p_y &= i\,\tau_{yx} + j\,\sigma_y + k\,\tau_{yz} \\ p_z &= i\,\tau_{zx} + j\,\tau_{zy} + k\,\sigma_z\,. \end{aligned}\right\} \quad (3.7)$$

The stress system is seen to require nine scalar quantities for its description. These nine quantities form a *stress tensor*. The set of nine components of the stress tensor is sometimes called the stress matrix:

$$\Pi = \begin{pmatrix} \sigma_x & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_y & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_z \end{pmatrix}. \quad (3.8)$$

The stress tensor and the corresponding matrix are symmetric, which means that two shearing stresses with subscripts which differ only in their order are equal. This can be demonstrated with reference to the equations of motion of an element of fluid. In general, its motion can be separated into an instantaneous translation and an instantaneous rotation, and only the latter needs to be considered for our purpose. Denoting the instantaneous angular acceleration of the element by $\dot\omega(\dot\omega_x, \dot\omega_y, \dot\omega_z)$, we can write for the rotation about the y-axis that

$$\dot\omega_y\,\mathrm{d}I_y = (\tau_{xz}\,\mathrm{d}y\,\mathrm{d}z)\,\mathrm{d}x - (\tau_{zx}\,\mathrm{d}x\,\mathrm{d}y)\,\mathrm{d}z = (\tau_{xz} - \tau_{zx})\,\mathrm{d}V$$

where $\mathrm{d}I_y$ is the elementary moment of inertia about the y-axis. Now the moment of inertia, $\mathrm{d}I$, is proportional to the fifth power of the linear dimensions of the parallelepiped, whereas its volume, $\mathrm{d}V$, is proportional to their third power. On contracting the element to a point, we notice that the left-hand side of the preceding equation vanishes faster than the right-hand side. Hence, ultimately,

$$\tau_{xy} - \tau_{yx} = 0$$

if $\dot\omega_y$ is not to become infinitely large. Analogous equations can be written for the remaining two axes, and the symmetry of the stress tensor can thus be demonstrated. It is clear from the argument that the stress tensor would cease to be symmetric if the fluid developed a local moment which was proportional to its volume, $\mathrm{d}V$. The latter may occur, for example, in an electrostatic field.

Owing to the fact that

$$\tau_{xy} = \tau_{yx}; \qquad \tau_{xz} = \tau_{zx}; \qquad \tau_{yz} = \tau_{zy}, \quad (3.9)$$

the stress matrix (3.8) contains only six different stress components and becomes symmetrical with respect to the principal diagonal:

$$\Pi = \begin{pmatrix} \sigma_x & \tau_{xy} & \tau_{xz} \\ \tau_{xy} & \sigma_y & \tau_{yz} \\ \tau_{xz} & \tau_{yz} & \sigma_z \end{pmatrix}. \quad (3.10)$$

The surface force per unit volume can be calculated from eqns. (3.6), (3.7), and (3.10) and becomes

$$\begin{aligned} P = i &\left(\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{xz}}{\partial z}\right) \cdots\cdots \text{comp. } x \\ + j &\left(\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{yz}}{\partial z}\right) \cdots\cdots \text{comp. } y \\ + k &\left(\underbrace{\frac{\partial \tau_{xz}}{\partial x}}_{\text{face } yz} + \underbrace{\frac{\partial \tau_{yz}}{\partial y}}_{\text{face } zx} + \underbrace{\frac{\partial \sigma_z}{\partial z}}_{\text{face } xy}\right) \cdots\cdots \text{comp. } z\,. \end{aligned} \quad (3.10\text{a})$$

Introducing the expression (3.10a) into the equation of motion (3.2), and resolving into components we have:

$$\left.\begin{aligned} \varrho\,\frac{\mathrm{D}u}{\mathrm{D}t} &= X + \left(\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{xz}}{\partial z}\right) \\ \varrho\,\frac{\mathrm{D}v}{\mathrm{D}t} &= Y + \left(\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{yz}}{\partial z}\right) \\ \varrho\,\frac{\mathrm{D}w}{\mathrm{D}t} &= Z + \left(\frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \sigma_z}{\partial z}\right). \end{aligned}\right\} \quad (3.11)$$

If the fluid is "frictionless" all shearing stresses vanish; only the normal stresses remain in the equation, and they are, moreover, equal. Their negative is defined as the pressure at the point x, y, z in the fluid:

$$\begin{aligned} \tau_{xy} &= \tau_{xz} = \tau_{yz} = 0 \\ \sigma_x &= \sigma_y = \sigma_z = -p\,. \end{aligned}$$

In such a *hydrostatic stress system*, the fluid pressure is equal to the arithmetical mean of the normal stresses taken with a negative sign. Since measurements which lead to the establishment of the thermodynamic equation of state are performed under such conditions, the fluid being at rest, this pressure is identical with the thermodynamic pressure in the equation of state. It is convenient to introduce the arithmetical mean of the three normal stresses — their sum being called the *trace* of the stress tensor — as a useful numerical quantitiy in the case of a *viscous fluid* in a state of motion also. It is still called the pressure, but its relation to the thermodynamic pressure requires further investigation. Although it then ceases to be equal to a particular stress which is normal to the surface, it has the property of being invariant with respect to transformations of the system of coordinates, as it is an invariant of the stress tensor, being defined as

$$\tfrac{1}{3}\,(\sigma_x + \sigma_y + \sigma_z) = -p\,. \quad (3.12)$$

We shall see in Sec. IIIe that it remains equal to the thermodynamic pressure in the absence of relaxation.

The system of the three equations (3.11) contains the six stresses $\sigma_x, \sigma_y, \sigma_z, \tau_{xy}, \tau_{xz}, \tau_{yz}$. The next task is to determine the relation between them and the strains so as to enable us to introduce the velocity components u, v, w into eqn. (3.11). Before giving this relation in Sec. IIId we shall investigate the system of strains in greater detail.

c. The rate at which a fluid element is strained in flow

When a continuous body of fluid is made to flow, every element in it is, generally speaking, displaced to a new position in the course of time. During this motion elements of fluid become strained, and since the motion of the fluid is completely determined when the velocity vector $\boldsymbol{w}$ is given as a function of time and position, $\boldsymbol{w} = \boldsymbol{w}(x,y,z,t)$, there exist kinematic relations between the components of the rate of strain and this function. The rate at which an element of fluid is strained depends on the *relative* motion of two points within it. We, therefore, consider the two neighbouring points A and B which are shown in Fig. 3.2. Owing to the presence of the velocity field, point A will be displaced to A′ in time $\mathrm{d}t$ by a distance $\boldsymbol{s} = \boldsymbol{w}\,\mathrm{d}t$; since, however, the velocity at B, imagined at a distance $\mathrm{d}\boldsymbol{r}$ from A, is different, point B will move to B′ displaced from B by $\boldsymbol{s} + \mathrm{d}\boldsymbol{s} = (\boldsymbol{w} + \mathrm{d}\boldsymbol{w})\,\mathrm{d}t$. More explicitly, if the components of velocity have the values u, v, w at A, then, at the neighbouring point B, the velocity components will be given to first order by the Taylor-series expansions

$$\left.\begin{aligned} u + \mathrm{d}u &= u + \frac{\partial u}{\partial x}\,\mathrm{d}x + \frac{\partial u}{\partial y}\,\mathrm{d}y + \frac{\partial u}{\partial z}\,\mathrm{d}z \\ v + \mathrm{d}v &= v + \frac{\partial v}{\partial x}\,\mathrm{d}x + \frac{\partial v}{\partial y}\,\mathrm{d}y + \frac{\partial v}{\partial z}\,\mathrm{d}z \\ w + \mathrm{d}w &= w + \frac{\partial w}{\partial x}\,\mathrm{d}x + \frac{\partial w}{\partial y}\,\mathrm{d}y + \frac{\partial w}{\partial z}\,\mathrm{d}z\,. \end{aligned}\right\} \qquad (3.13)$$

Thus, the relative motion of point B with respect to A is described by the following matrix of nine partial derivatives of the local velocity field

$$\begin{pmatrix} \dfrac{\partial u}{\partial x} & \dfrac{\partial u}{\partial y} & \dfrac{\partial u}{\partial z} \\ \dfrac{\partial v}{\partial x} & \dfrac{\partial v}{\partial y} & \dfrac{\partial v}{\partial z} \\ \dfrac{\partial w}{\partial x} & \dfrac{\partial w}{\partial y} & \dfrac{\partial w}{\partial z} \end{pmatrix} \qquad (3.13\,\mathrm{a})$$

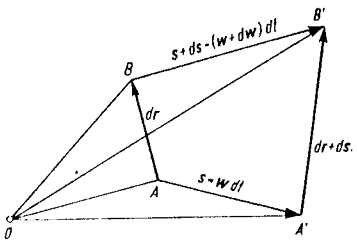

Fig. 3.2. Relative displacement

It is convenient to rearrange the expressions for the relative velocity components $\mathrm{d}u$, $\mathrm{d}v$, $\mathrm{d}w$ from eqn. (3.13) to the form

$$\left.\begin{aligned} \mathrm{d}u &= (\dot\varepsilon_x\,\mathrm{d}x + \dot\varepsilon_{xy}\,\mathrm{d}y + \dot\varepsilon_{xz}\,\mathrm{d}z) + (\eta\,\mathrm{d}z - \zeta\,\mathrm{d}y) \\ \mathrm{d}v &= (\dot\varepsilon_{yx}\,\mathrm{d}x + \dot\varepsilon_{y}\,\mathrm{d}y + \dot\varepsilon_{yz}\,\mathrm{d}z) + (\zeta\,\mathrm{d}x - \xi\,\mathrm{d}z) \\ \mathrm{d}w &= (\dot\varepsilon_{zx}\,\mathrm{d}x + \dot\varepsilon_{zy}\,\mathrm{d}y + \dot\varepsilon_{z}\,\mathrm{d}z) + (\xi\,\mathrm{d}y - \eta\,\mathrm{d}x)\,, \end{aligned}\right\} \qquad (3.14)$$

it being easy to verify that the new symbols have the following meanings

$$\dot\varepsilon_{ij} = \begin{pmatrix} \dot\varepsilon_x & \dot\varepsilon_{xy} & \dot\varepsilon_{xz} \\ \dot\varepsilon_{yx} & \dot\varepsilon_y & \dot\varepsilon_{yz} \\ \dot\varepsilon_{zx} & \dot\varepsilon_{zy} & \dot\varepsilon_z \end{pmatrix} = \begin{pmatrix} \dfrac{\partial u}{\partial x}\,; & \dfrac{1}{2}\left(\dfrac{\partial v}{\partial x} + \dfrac{\partial u}{\partial y}\right); & \dfrac{1}{2}\left(\dfrac{\partial w}{\partial x} + \dfrac{\partial u}{\partial z}\right); \\ \dfrac{1}{2}\left(\dfrac{\partial u}{\partial y} + \dfrac{\partial v}{\partial x}\right); & \dfrac{\partial v}{\partial y}\,; & \dfrac{1}{2}\left(\dfrac{\partial w}{\partial y} + \dfrac{\partial v}{\partial z}\right); \\ \dfrac{1}{2}\left(\dfrac{\partial u}{\partial z} + \dfrac{\partial w}{\partial x}\right); & \dfrac{1}{2}\left(\dfrac{\partial v}{\partial z} + \dfrac{\partial w}{\partial y}\right); & \dfrac{\partial w}{\partial z} \end{pmatrix} \qquad (3.15\,\mathrm{a})$$

and

$$\xi = \frac{1}{2}\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right); \quad \eta = \frac{1}{2}\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right); \quad \zeta = \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right). \qquad (3.15\,\mathrm{b})$$

It is noted that the matrix $\dot\varepsilon_{ij}$ is symmetric, so that

$$\dot\varepsilon_{yx} = \dot\varepsilon_{xy}\,; \qquad \dot\varepsilon_{xz} = \dot\varepsilon_{zx}\,; \qquad \dot\varepsilon_{zy} = \dot\varepsilon_{yz}\,, \qquad (3.15\,\mathrm{c})$$

and that ξ, η, ζ are related to the components of the vector

$$\boldsymbol{\omega} = \operatorname{curl} \boldsymbol{w} \qquad (3.15\,\mathrm{d})$$

Each of the new terms can be given a kinematic interpretation, and we now proceed to obtain it.

Since we concentrate our attention on the immediate neighbourhood of point A, and since interest is centred on the motion of B relative to A, we shall place point A at the origin, and interpret $\mathrm{d}x$, $\mathrm{d}y$, $\mathrm{d}z$ as the coordinates of point B in a Cartesian system of coordinates. In this manner, the expressions in eqns. (3.14) will define a field of relative velocities in which the components $\mathrm{d}u$, $\mathrm{d}v$, $\mathrm{d}w$ are linear functions of the space coordinates. In order to understand the meaning of the different terms in the matrix (3.15a) and in eqns. (3.15b), we proceed to interpret them one by one.

The diagram in Fig. 3.3 represents the field of relative velocities when all terms except $\partial u/\partial x$ vanish on the assumption that $\partial u/\partial x > 0$. The relative velocity of any point B with respect to A is now

$$\mathrm{d}u = \left(\frac{\partial u}{\partial x}\right)\mathrm{d}x\,,$$

and the field consists of planes $x = \text{const}$ which displace themselves uniformly with a velocity which is proportional to the distance $\mathrm{d}x$ away from the plane $x = 0$. An elementary parallelepiped with A and B at its vertices placed in such a velocity field will be distorted in extension, its face BC receding from AD with an increasing

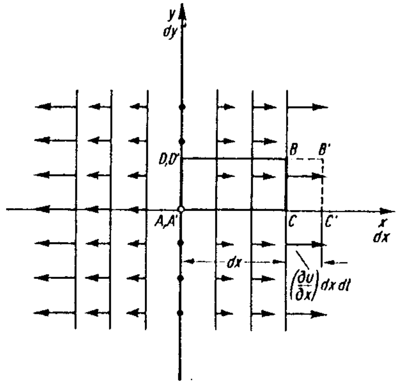

Fig. 3.3. Local distortion of fluid element when $\partial u/\partial x > 0$ with all other terms being equal to zero; uniform extension in the x-direction

velocity. Thus $\dot{\varepsilon}_x$ represents the rate of *elongation* in the x-direction suffered by the element. Similarly, the additive terms $\dot{\varepsilon}_y = \partial v/\partial y$ and $\dot{\varepsilon}_z = \partial w/\partial z$ describe the rate of elongation in the y- and z-directions, respectively.

It is now easy to visualize the distortion imparted to a fluid element by the simultaneous action of all three diagonal elements of matrices (3.13a) or (3.15a). The element expands in all three directions, and the change in the length of its three sides produces a change in volume at a relative rate

$$\dot{e} = \frac{\left\{dx + \frac{\partial u}{\partial x}\, dx\, dt\right\}\left\{dy + \frac{\partial v}{\partial y}\, dy\, dt\right\}\left\{dz + \frac{\partial w}{\partial z}\, dz\, dt\right\} - dx\, dy\, dz}{dx\, dy\, dz\, dt}$$

$$= \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = \operatorname{div} \boldsymbol{w}\,, \tag{3.16}$$

to first order in the derivatives. During this distortion, however, the shape of the element, described by the angles at its vertices, remains unchanged, since all right angles continue to be that way. Thus $\dot{e}$ describes the local, instantaneous *volumetric dilatation* of a fluid element. When the fluid is incompressible, $\dot{e} = 0$, as must be expected. In a compressible fluid the continuity equation (3.1) shows that

$$\dot{e} = \operatorname{div} \boldsymbol{w} = -\frac{1}{\varrho}\frac{D\varrho}{Dt}\,, \tag{3.17}$$

that is that the volumetric dilatation, the relative change in volume, is equal to the negative of the relative rate of change in the local density.

The relative velocity field presents a different appearance when one of the off-diagonal terms of matrix (3.13a), for example $\partial u/\partial y$, has a non-vanishing, say positive, value. The corresponding field, sketched in Fig. 3.4, is one of pure shear strain. A rectangular element of fluid centred on A now distorts into a parallelogram as indicated in the diagram. The original right angle at A changes at a rate measured by the angle $\gamma_{xy} = [(\partial u/\partial y)\, dy\, dt]/dy$, that is at a rate $\partial u/\partial y$. When both $\partial u/\partial y$

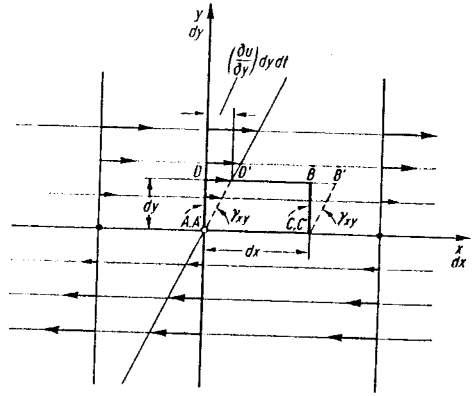

Fig. 3.4. Local distortion of fluid element when $\partial u/\partial y > 0$ with all other terms being equal to zero; uniform shear deformation.

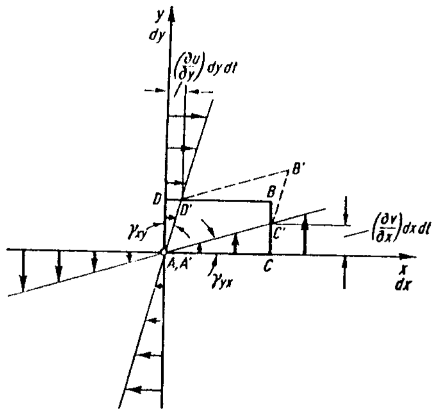

Fig. 3.5. Local distortion of fluid element when

$$\dot{\varepsilon}_{xy} = \dot{\varepsilon}_{yx} = \tfrac{1}{2}\{(\partial u/\partial y) + (\partial v/\partial x)\} > 0$$

with all other terms being equal to zero; distortion in shape. (The diagram has been drawn for $\partial u/\partial y = \partial v/\partial x$)

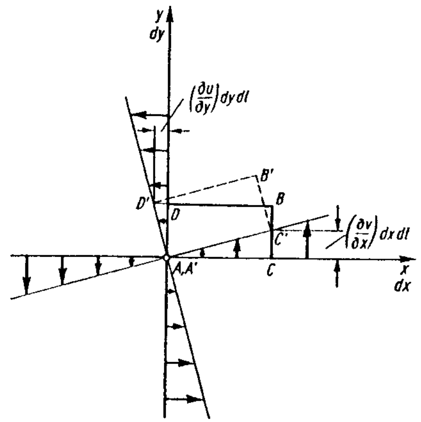

Fig. 3.6. Local distortion of fluid element when

$$\zeta = \tfrac{1}{2}\{(\partial v/\partial x) - (\partial u/\partial y)\} \neq 0\,;$$

instantaneous rigid-body rotation

and $\partial v/\partial x$ have positive nonvanishing values, the right angle at A will distort owing to the superposition of two motions, the state of affairs being illustrated in Fig. 3.5. It is clear that the right angle at A now distorts at twice the rate

$$\dot{\varepsilon}_{yx} = \dot{\varepsilon}_{xy} = \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)$$

described by two of the off-diagonal terms of matrix (3.15a). In general, the three off-diagonal terms $\dot{\varepsilon}_{xy} = \dot{\varepsilon}_{yx}$, $\dot{\varepsilon}_{xz} = \dot{\varepsilon}_{zx}$, and $\dot{\varepsilon}_{zy} = \dot{\varepsilon}_{yz}$ describe the rate of distortion of a right angle located in a plane normal to the axis the index of which does not appear as a subscript. The distortion is volume-preserving and affects only the shape of the element.

Circumstances are again different in the particular case when $\partial u/\partial y = -\partial v/\partial x$ illustrated in Fig. 3.6. From the preceding considerations and from the fact that now $\dot{\varepsilon}_{xy} = 0$ we can infer at once that the right angle at A remains undistorted. This is also clear from the diagram which shows that the fluid element rotates with respect to the reference point A. *Instantaneously*, this rotation occurs without distortion and can be described as a rigid-body rotation. The instantaneous angular velocity of this rotation is

$$\frac{(\partial v/\partial x)\,\mathrm{d}x\,\mathrm{d}t}{\mathrm{d}x\,\mathrm{d}t} = \frac{\partial v}{\partial x} \quad \text{or} \quad = -\frac{\partial u}{\partial y}\,.$$

It is now easy to see that the component ζ of $\frac{1}{2}$ curl $\boldsymbol{w}$ from eqn. (3.15b), known as the vorticity of the velocity field, represents the angular velocity of this instantaneous rigid-body rotation, and that

$$\frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) \neq 0\,.$$

In the more complex case when $(\partial v/\partial x) \neq -(\partial u/\partial y)$, the element of fluid rotates and its shape is distorted simultaneously. We can still interpret the term

$$\dot{\varepsilon}_{xy} = \dot{\varepsilon}_{yx} = \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)$$

as describing the rate of distortion in shape, the term

$$\zeta = \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)$$

describing the rate at which the element of fluid participates in a rigid-body rotation.

The linearity of eqns. (3.13) or of the entirely equivalent eqns. (3.14) signifies that the most general case arises by a superposition of the simple cases just described. Therefore, if attention is fixed on two neighbouring points A and B in a body of fluid which sustains a continuous velocity field $\boldsymbol{w}(x,y,z)$, the motion of an element of fluid surrounding these two points can be uniquely decomposed into four component motions:

(a) A pure translation described by the velocity components u, v, w of $\boldsymbol{w}$.
(b) A rigid-body rotation described by the components ξ, η, ζ of $\frac{1}{2}$ curl $\boldsymbol{w}$.
(c) A volumetric dilatation described by $e = \operatorname{div} \boldsymbol{w}$, the linear dilatations in the direction of the axes being described by $\dot{\varepsilon}_x, \dot{\varepsilon}_y$ and $\dot{\varepsilon}_z$, respectively.
(d) A distortion in shape described by the components $\dot{\varepsilon}_{xy}$ etc with mixed indices.

Only the last two motions produce an intrinsic deformation of a fluid element surrounding the reference point A, the first two causing a mere, general, displacement of its location.

The elements of matrix (3.15a) constitute the components of a symmetric tensor known as the *rate-of-strain tensor*; its mathematical properties are analogous to those of the equally symmetric stress tensor. It is known from the theory of elasticity [3, 7] or from general considerations of tensor algebra [11] that with every symmetric tensor it is possible to associate three mutually orthogonal *principal axes* which determine three mutually orthogonal principal planes that is a privileged Cartesian system of coordinates. In this system of coordinates, the stress vector or the instantaneous motion in any one of the principal planes is normal to it, that is, parallel to one of the axes. When such a special system of coordinates is used, the matrices (3.10) or (3.15a) retain their diagonal terms only. Denoting the values of the respective components by symbols with bars, we would be dealing with the matrices

$$\begin{pmatrix} \bar{\sigma}_x & 0 & 0 \\ 0 & \bar{\sigma}_y & 0 \\ 0 & 0 & \bar{\sigma}_z \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \bar{\dot{\varepsilon}}_x & 0 & 0 \\ 0 & \bar{\dot{\varepsilon}}_y & 0 \\ 0 & 0 & \bar{\dot{\varepsilon}}_z \end{pmatrix} \tag{3.18}$$

It should, finally, be remembered that such a transformation of coordinates does not affect the sum of the diagonal terms, so that

$$\sigma_x + \sigma_y + \sigma_z = \bar{\sigma}_x + \bar{\sigma}_y + \bar{\sigma}_z\,, \tag{3.19a}$$

and

$$\dot{\varepsilon}_x + \dot{\varepsilon}_y + \dot{\varepsilon}_z = \bar{\dot{\varepsilon}}_x + \bar{\dot{\varepsilon}}_y + \bar{\dot{\varepsilon}}_z \quad (= e = \operatorname{div} \boldsymbol{w})\,, \tag{3.19b}$$

because they constitute invariants of the tensors, as already intimated earlier. Viewed in such two systems of coordinates (both denoted by bars), an element

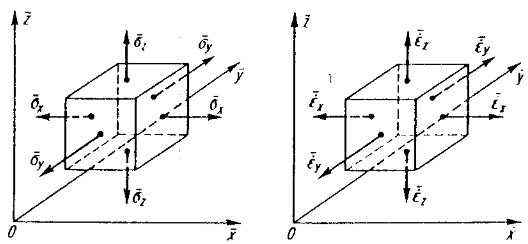

Fig. 3.7. Principial axes for stress and rate of strain

of fluid is stressed in three mutually perpendicular directions, and its faces are displaced instantaneously also in three mutually perpendicular directions, as suggested by Figs. 3.7a and b. This does not, of course, mean that there exist no shearing stresses in other planes or that the shape of the element remains undistorted.

d. Relation between stress and rate of deformation

It should, perhaps, be stressed once more that the equations which relate the surface forces to the flow field must be obtained by a perceptive interpretation of experimental results and that our interest is restricted to isotropic and Newtonian fluids. The considerations of the preceding section provided us with a useful mathematical framework which allows us now to state the requirements suggested by experiments in a somewhat more precise form.

When the fluid is at rest, it develops a uniform field of hydrostatic stress (negative pressure $-p$) which is identical with the thermodynamic pressure. When the fluid is in motion, the equation of state still determines a pressure at every point ("principle of local state" [4]), and it is convenient to consider the deviatoric normal stresses

$$\sigma_x' = \sigma_x + p\,; \qquad \sigma_y' = \sigma_y + p\,; \qquad \sigma_z' = \sigma_z + p\,; \tag{3.20}$$

together with the unchanged shearing stresses. The six quantities so obtained constitute a symmetric stress tensor the existence of which is due to the motion because at rest all its components vanish identically. From what has been said before it follows that the components of this deviatoric tensor are created solely by the components of the rate-of-strain tensor, that is to the exclusion of the components u, v, w of velocity as well as of the components ξ, η, ζ of vorticity. This is equivalent to saying that the instantaneous translation [component motion (a)] as well as the instantaneous rigid-body rotation [component motion (b)] of an element of fluid produce no surface forces on it in addition to the existing components of hydrostatic pressure. The preceding statement, evidently, merely represents a precise local formulation of what we expect to observe when a finite body of fluid performs a general motion which is indistinguishable from that of an equivalent rigid body. We thus conclude that the expressions for the components $\sigma_x', \sigma_y', \ldots, \tau_{zx}$ of the deviatoric stress tensor can contain in them only the velocity gradients $\partial u/\partial x, \ldots, \partial w/\partial z$ in appropriate combinations which we now proceed to determine. These relations are postulated to be linear; they must remain unchanged by a rotation of the system of coordinates or by an interchange of axes to ensure isotropy. Isotropy also requires that at every point in the continuum, the principal axes of the stress tensor must coincide with the principal axes of the rate-of-strain tensor, for, otherwise, a preferred direction would be introduced. The simplest way to achieve our aim is to select an arbitrary point in the continuum and to imagine that the local system of coordinates $\bar{x}, \bar{y}, \bar{z}$ has been provisionally so chosen as to coincide with the three common principal axes of the two tensors. The components of the velocity field in this system of coordinates are denoted by $\bar{u}, \bar{v}, \bar{w}$.

It is now clear that isotropy can be secured only if each one of the three normal stresses $\bar{\sigma}_x', \bar{\sigma}_y', \bar{\sigma}_z'$ is made to depend on the component of rate of strain the direction

of which conicides with it and on the sum of the three, each with a different factor of proportionality. Thus we record, directly in terms of the space-derivatives, that

$$\left.\begin{aligned}
\bar{\sigma}_x' &= \lambda\left(\frac{\partial \bar{u}}{\partial \bar{x}} + \frac{\partial \bar{v}}{\partial \bar{y}} + \frac{\partial \bar{w}}{\partial \bar{z}}\right) + 2\,\mu\,\frac{\partial \bar{u}}{\partial \bar{x}}\\
\bar{\sigma}_y' &= \lambda\left(\frac{\partial \bar{u}}{\partial \bar{x}} + \frac{\partial \bar{v}}{\partial \bar{y}} + \frac{\partial \bar{w}}{\partial \bar{z}}\right) + 2\,\mu\,\frac{\partial \bar{v}}{\partial \bar{y}}\\
\bar{\sigma}_z' &= \lambda\left(\frac{\partial \bar{u}}{\partial \bar{x}} + \frac{\partial \bar{v}}{\partial \bar{y}} + \frac{\partial \bar{w}}{\partial \bar{z}}\right) + 2\,\mu\,\frac{\partial \bar{w}}{\partial z}\,.
\end{aligned}\right\} \tag{3.21}$$

The quantities u, v, w and ξ, η, ζ do not appear in these expressions for the reasons just explained. In each expression, the last term represents the appropriate rate of linear dilatation, that is, in essence, a change in shape, and the first term represents the volumetric dilatation, that is the rate of change in volume, in essence, a change in density. The factors 2 in the last terms are not essential, being merely convenient to facilitate the interpretation, as we shall see later. The factors of proportionality, μ and λ, two in all, must be the same in each of the three preceding equations to secure isotropy. It is easy to see that an interchange between any two axes, that is an interchange of any of the three pairs of quantities: $(\bar{u}, \bar{x})$, $(\bar{v}, \bar{y})$, $(\bar{w}, \bar{z})$ leaves the set of relations invariant, as they must be in an isotropic medium. Moreover, the preceding is the only combination of spatial gradients which possesses the required properties. If the reader cannot see this directly, he may consult a more rigorous proof in a treatise on tensor algebra (or e. g. [11] p. 89).

The relations in eqns. (3.21) can be re-written to apply in an arbitrary system of coordinates by performing a general rotation with the aid of the appropriate linear transformation formulae. We shall refrain from putting down the detailed steps because, though tedious if performed directly, they are quite straightforward. They become simple if tensor calculus is used. The approriate direct formulae may be found in refs. [3, 6, 7], whereas their tensorial counterparts are given in ref. [11]. Such a derivation would show that

$$\left.\begin{aligned}
\sigma_x' &= \lambda\,\mathrm{div}\,\boldsymbol{w} + 2\,\mu\,\frac{\partial u}{\partial x}\\
\sigma_y' &= \lambda\,\mathrm{div}\,\boldsymbol{w} + 2\,\mu\,\frac{\partial v}{\partial y}\\
\sigma_z' &= \lambda\,\mathrm{div}\,\boldsymbol{w} + 2\,\mu\,\frac{\partial w}{\partial z}\,;
\end{aligned}\right\} \tag{3.22a}$$

$$\left.\begin{aligned}
\tau_{xy} &= \tau_{yx} = \mu\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)\\
\tau_{yz} &= \tau_{zy} = \mu\left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}\right)\\
\tau_{zx} &= \tau_{xz} = \mu\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right)\,;
\end{aligned}\right\} \tag{3.22b}$$

where div w has been used for brevity. The reader may notice the regularity with which the indices x, y, z, the components u, v, w, and the coordinates x, y, z are permuted†.

Applying these equations to the simple case represented in Fig. 1.1, we recover eqn. (1.2) and so confirm that the preceding more general relation reduces to Newton's law of friction in the case of simple shear and does, therefore, constitute its proper generalization. At the same time, we identify the factor μ with the viscosity of the fluid, amply discussed in Sec 1b, and, incidentally, justify the factor 2 previously inserted into eqns. (3.21). The physical significance of the second factor, λ, requires further discussion, but we note that it plays no part in an incompressible fluid when div $w = 0$; it then disappears from the equations altogether, and so is seen to be important for compressible fluids only.

e. Stokes's hypothesis

Although the problem that we are about to discuss has arisen more than a century and a half ago, the physical interpretation of the second factor, λ, in eqns. (3.21) or (3.22a, b) and for flows in which div w does not vanish identically, is still being disputed, even though the *value* which should be given to it *in the working equations* is not. This numerical value is determined with the aid of a hypothesis advanced by G. G. Stokes in 1845 [13]. Without, for the moment, concerning ourselves with the physical reasons which justify *Stokes's hypothesis*, we first state that according to it, it is necessary to assume

$$3\lambda + 2\mu = 0\,, \quad \text{or} \quad \lambda = -\frac{2}{3}\mu\,. \tag{3.23}$$

This relates the value of the factor λ to the viscosity, μ, of the compressible fluid and reduces the number of properties which characterize the field of stresses in a flowing compressible fluid from two to one, that is to the same number as is required for an incompressible fluid.

Substituting this value into eqns. (3.22a), we obtain the normal components of deviatoric stress:

$$\left.\begin{aligned}
\sigma_x' &= -\frac{2}{3}\mu\,\mathrm{div}\,w + 2\mu\frac{\partial u}{\partial x}\\
\sigma_y' &= -\frac{2}{3}\mu\,\mathrm{div}\,w + 2\mu\frac{\partial v}{\partial y}\\
\sigma_z' &= -\frac{2}{3}\mu\,\mathrm{div}\,w + 2\mu\frac{\partial w}{\partial z}\,,
\end{aligned}\right\} \tag{3.24}$$

† The above set of six equations can be contracted to a single one in Cartesian-tensor notation (with Einstein's summation convention):

$$\sigma_{ij}' = \lambda\,\delta_{ij}\frac{\partial v_k}{\partial x_k} + \mu\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right), \quad (i, j, k = 1, 2, 3)$$

where the Kronecker delta $\delta_{ij} = 0$ for $i \neq j$ and $\delta_{ij} = 1$ for $i = j$.

the shearing stresses remaining unchanged. Making use of eqns. (3.20), we obtain the so-called *constitutive equation* for an isotropic, Newtonian fluid

$$\left.\begin{aligned}
\sigma_x &= -p - \frac{2}{3}\mu\,\mathrm{div}\,w + 2\mu\frac{\partial u}{\partial x}\\
\sigma_y &= -p - \frac{2}{3}\mu\,\mathrm{div}\,w + 2\mu\frac{\partial v}{\partial y}\\
\sigma_z &= -p - \frac{2}{3}\mu\,\mathrm{div}\,w + 2\mu\frac{\partial w}{\partial z}
\end{aligned}\right\} \tag{3.25a}$$

$$\left.\begin{aligned}
\tau_{xy} &= \tau_{yx} = \mu\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)\\
\tau_{yz} &= \tau_{zy} = \mu\left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}\right)\\
\tau_{zx} &= \tau_{xz} = \mu\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right)
\end{aligned}\right\} \tag{3.25b}$$

in its final form, noting that p represents the local thermodynamic pressure†.

Regarded as a pure hypothesis, or even guess, eqn. (3.23) can certainly be accepted on the ground that the working equations which result from the substitution of eqns. (3.25a, b) into (3.11) have been subjected to an unusually large number of experimental verifications, even under quite extreme conditions, as the reader will concede after having studied this book. Thus, even if it should not represent the state of affairs exactly, it certainly constitutes an excellent approximation.

Since the deviatoric components are the only ones which arise in motion, they represent those components of stress which produce dissipation in an isothermal flow, there being further dissipation in a temperature field due to thermal conduction, Chap. XII. Furthermore, since the factor λ occurs only in the normal components $\sigma_x', \sigma_y', \sigma_z'$ which also contain the thermodynamic pressure, eqns. (3.20), it becomes clear that the physical significance of λ is connected with the mechanism of dissipation when the volume of the fluid element is changed at a finite rate as well as with the relation between the total stress tensor and thermodynamic pressure.

f. Bulk viscosity and thermodynamic pressure

We now revert to the general discussion, without necessarily accepting the validity of Stokes's hypothesis, but confine it to the case when no shearing stresses are involved, because their physical significance and origin is clear. Consequently,

† In the compact tensorial notation we would write

$$\sigma_{ij} = -p\,\delta_{ij} + \mu\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} - \frac{2}{3}\delta_{ij}\frac{\partial v_k}{\partial x_k}\right) \quad (i, j, k = 1, 2, 3)\,.$$

we consider a fluid system, say the sphere shown in Fig. 3.8a which is subjected to a uniform normal stress, $\overline{\sigma}$, on its boundary. In the absence of motion $\overline{\sigma}$ is obviously equal and opposite in sign to the thermodynamic pressure, p. Taking the sum of the three equations (3.21) and utilizing eqns. (3.20), we find that

$$\overline{\sigma} = -p + (\lambda + \frac{2}{3}\mu)\,\mathrm{div}\,\boldsymbol{w}\,, \tag{3.26}$$

and notice that our equations reflect this fact, as already pointed out earlier. Now, the question poses itself as to what this relation should be in a general flow field.

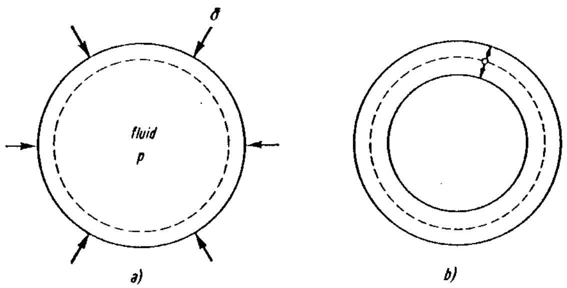

Fig. 3.8. Quasistatic compression and oscillatory motion of a spherical mass of fluid

When the system is compressed quasistatically and reversibly, we again recover the previous case because then div $\boldsymbol{w} \to 0$ asymptotically. We note that in such cases the rate at which work is performed in a thermodynamically reversible process per unit volume becomes

$$\dot{W} = p\,\mathrm{div}\,\boldsymbol{w} \tag{3.26a}$$

which is the same as

$$\dot{W} = p\,\frac{\mathrm{d}V}{\mathrm{d}t} \tag{3.26b}$$

in the notation customary in thermodynamics.

When div $\boldsymbol{w}$ is finite, and the fluid is compressed, expanded or made to oscillate, at a finite rate, equality between $\overline{\sigma}$ and $-p$ persists only if the coefficient

$$\mu' = \lambda + \frac{2}{3}\mu \tag{3.27}$$

vanishes identically (Stokes's hypothesis); otherwise it does not. If $\mu' \neq 0$, the oscillatory motion of a spherical system, Fig. 3.8b, would produce dissipation, even if the temperature remained constant throughout the bulk of the gas. The same would be true in the case of expansion or compression at a finite rate. For this reason, the coefficient μ' is called the *bulk viscosity* of the fluid: it represents that property, like the shear viscosity μ for deformation in shape, which is responsible for energy dissipation in a fluid of uniform temperature during a change in volume at a finite rate. The bulk viscosity would thus constitute a second property of a compressible, isotropic, Newtonian fluid needed to determine its constitutive equation and would have to be measured in addition to μ. It is evident that

$$\mu' = 0 \quad \text{implies} \quad p = -\overline{\sigma}$$
$$\mu' \neq 0 \quad \text{implies} \quad p \neq -\overline{\sigma}\,.$$

Thus the acceptance of Stokes's hypothesis is equivalent to the assumption that the thermodynamic pressure p is equal the one-third of the invariant sum of normal stresses even in cases when compression or expansion proceeds at a finite rate. Furthermore, it is also equivalent to the assumption that the oscillatory motion of a large spherical system would be reversible if it were isothermal. More detailed considerations in terms of the concepts of thermodynamics as it applies to irreversible processes in continuous systems can be found in the works of J. Meixner [8], I. Prigogine [12] and S. R. de Groot and P. Mazur [1].

In order to determine under what conditions the bulk viscosity of a compressible fluid vanishes, it is necessary to have recourse to experiment or to the methods of statistical thermodynamics which permit us to calculate transport coefficients from first principles. The direct measurement of bulk viscosity is very difficult to perform, and no definitive results are in existence. Statistical methods for dense gases or liquids have not yet been developed to a point which would allow us to make a complete statement on the subject. It appears, however, that the bulk viscosity vanishes identically in gases of low density, that is under conditions when only binary collisions of molecules need to be taken into account. In dense gases, the numerical value of bulk viscosity appears to be very small. This means that eqns. (3.26a,b) continue to describe the work in a continuous system in the absence of shear to an excellent degree of approximation and that dissipation at constant temperature, even in the general case, occurs only through the intervention of the deviatoric stresses. Thus, once again, we are led to Stokes's hypothesis and so to eqn. (3.26). This conclusion does not extend to fluids which are capable of undergoing relaxation processes by virtue of a local departure from a state of chemical equilibrium [1,8]. Such relaxation processes occur, for example, when a chemical reaction can take place, or, in gases of complex structure, when a comparatively slow transfer of energy between the translational and rotational degrees of freedom on the one hand, and the vibrational degrees of freedom on the other, becomes possible. Thus when relaxation processes are possible, the thermodynamic pressure is no longer equal to one-third of the trace of the stress tensor.

It is sometimes argued that the adoption of Stokes's hypothesis, that is the supposition that the bulk viscosity of Newtonian fluid vanishes, does not accord with our intuitive feeling that a sphere of fluid whose boundary oscillates so that there is a cyclic sequence of compression and expansion, Fig. 3.8b, would dissipate no energy. This would, indeed, be the case, as is easily seen from the preceding argument, because the dissipative part of the stress field vanishes under such conditions. It must, however, not be forgotten that such a conclusion is valid only if the temperature of the sphere of gas were to be kept constant during the oscillation throughout the whole volume. Normally this is impossible. Consequently, an oscillating sphere of gas will soon develop a temperature field and energy will be dissipated down the existing temperature gradients [5].

g. The Navier-Stokes equations

With the aid of eqns. (3.20) the non-viscous pressure terms can be separated in the equation of motion (3.11) so that they become

$$\left.\begin{aligned}
\varrho \frac{\mathrm{D}u}{\mathrm{D}t} &= X - \frac{\partial p}{\partial x} + \left(\frac{\partial \sigma_x'}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{xz}}{\partial z}\right)\\
\varrho \frac{\mathrm{D}v}{\mathrm{D}t} &= Y - \frac{\partial p}{\partial y} + \left(\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y'}{\partial y} + \frac{\partial \tau_{yz}}{\partial z}\right)\\
\varrho \frac{\mathrm{D}w}{\mathrm{D}t} &= Z - \frac{\partial p}{\partial z} + \left(\frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \sigma_z'}{\partial z}\right).
\end{aligned}\right\} \tag{3.28}$$

Introducing the constitutive relation from eqns. (3.24) we obtain the resultant surface force in terms of the velocity components, e. g. for the x-direction we obtain with the aid of eqn. (3.10a):

$$P_x = \frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{xz}}{\partial z} = -\frac{\partial p}{\partial x} + \frac{\partial \sigma_x'}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{xz}}{\partial z}$$

$$P_x = -\frac{\partial p}{\partial x} + \frac{\partial}{\partial x}\left[2\,\mu \frac{\partial u}{\partial x} - \frac{2}{3}\,\mu \operatorname{div} \mathbf{w}\right] + \frac{\partial}{\partial y}\left[\mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)\right] + \frac{\partial}{\partial z}\left[\mu\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right)\right]$$

and corresponding expressions for the y- and z-components. In the general case of a compressible flow, the viscosity μ must be regarded as dependent on the space coordinates, because μ varies considerably with temperature (Tables 1.2 and 12.1), and the changes in velocity and pressure together with the heat due to friction bring about considerable temperature variations. The temperature dependence of viscosity μ(T) must be obtained from experiments (*cf.* Sec. XIIIa).

If these expressions are introduced into the fundamental equations (3.11), we obtain

$$\begin{aligned}
\varrho \frac{\mathrm{D}u}{\mathrm{D}t} &= X - \frac{\partial p}{\partial x} + \frac{\partial}{\partial x}\left[\mu\left(2\frac{\partial u}{\partial x} - \frac{2}{3}\operatorname{div}\mathbf{w}\right)\right] + \frac{\partial}{\partial y}\left[\mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)\right] + \frac{\partial}{\partial z}\left[\mu\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right)\right]\\
\varrho \frac{\mathrm{D}v}{\mathrm{D}t} &= Y - \frac{\partial p}{\partial y} + \frac{\partial}{\partial y}\left[\mu\left(2\frac{\partial v}{\partial y} - \frac{2}{3}\operatorname{div}\mathbf{w}\right)\right] + \frac{\partial}{\partial z}\left[\mu\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right)\right] + \frac{\partial}{\partial x}\left[\mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)\right]\\
\varrho \frac{\mathrm{D}w}{\mathrm{D}t} &= Z - \frac{\partial p}{\partial z} + \frac{\partial}{\partial z}\left[\mu\left(2\frac{\partial w}{\partial z} - \frac{2}{3}\operatorname{div}\mathbf{w}\right)\right] + \frac{\partial}{\partial x}\left[\mu\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right)\right] + \frac{\partial}{\partial y}\left[\mu\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right)\right].
\end{aligned}$$

(3.29a, b, c)†

These very well known differential equations form the basis of the whole science of fluid mechanics. They are usually referred to as the Navier-Stokes equations.

† In indicial notation:

$$\varrho\left(\frac{\partial v_i}{\partial t} + v_j \frac{\partial v_i}{\partial x_j}\right) = X_i - \frac{\partial p}{\partial x_i} + \frac{\partial}{\partial x_j}\left\{\mu\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} - \frac{2}{3}\,\delta_{ij}\,\frac{\partial v_k}{\partial x_k}\right)\right\} \quad (i, j, k = 1, 2, 3)\,.$$

It is necessary to include here the equation of continuity which, as seen from eqn. (3.1), assumes the following form for compressible flow:

$$\frac{\partial \varrho}{\partial t} + \frac{\partial(\varrho\, u)}{\partial x} + \frac{\partial(\varrho\, v)}{\partial y} + \frac{\partial(\varrho\, w)}{\partial z} = 0\,. \tag{3.30}$$

The above equations do not give a complete description of the motion of a compressible fluid because changes in pressure and density effect temperature variations, and principles of thermodynamics must, therefore, once more enter into the considerations. From thermodynamics we obtain, in the first place, the characteristic equation (equation of state) which combines pressure, density, and temperature, and which for a perfect gas has the form

$$p - \varrho\, R\, T = 0\,, \tag{3.31}$$

with R denoting the gas constant and T denoting the absolute temperature. Secondly, if the process is not isothermal, it is further necessary to make use of the energy equation which draws up a balance between heat and mechanical energy (First Law of Thermodynamics), and which furnishes a differential equation for the temperature distribution. The energy equation will be discussed in greater detail in Chap. XII. The final equation of the system is given by the empirical viscosity law $\mu(T)$, its dependence on pressure being, normally, neglected. In all, if the forces X, Y, Z are considered given, there are seven equations for the seven variables $u, v, w, p, \varrho, T, \mu$.

For isothermal processes these reduce to five equations (3.29a, b, c), (3.30) and (3.31) for the five unknowns u, v, w, p, ϱ.

Incompressible flow: The above system of equations becomes further simplified in the case of incompressible fluids (ϱ = const) even if the temperature is not constant. First, as already shown in eqn. (3.1a), we have div $\mathbf{w} = 0$. Secondly, since temperature variations are, generally speaking, small in this case, the viscosity may be taken to be constant†.

The equation of state as well as the energy equation become superfluous as far as the calculation of the field of flow is concerned. The field of flow can now be considered independently from the equations of thermodynamics. The equations of motion (3.29a, b, c) and (3.30) can be simplified and, if the acceleration terms are written out fully, they assume the following form:

$$\left.\begin{aligned}
\varrho\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}\right) &= X - \frac{\partial p}{\partial x} + \mu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right)\\
\varrho\left(\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z}\right) &= Y - \frac{\partial p}{\partial y} + \mu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2}\right)\\
\varrho\left(\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z}\right) &= Z - \frac{\partial p}{\partial z} + \mu\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2}\right)
\end{aligned}\right\} \tag{3.32a, b, c}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0\,. \tag{3.33}$$

† This condition is more nearly satisfied in gases than in liquids.

With known body forces there are four equations for the four unknowns u, v, w, p.

If vector notation is used the simplified Navier-Stokes equations for incompressible flow, eqns. (3.32a, b, c), can be shortened to

$$\varrho \frac{Dw}{Dt} = F - \operatorname{grad} p + \mu \nabla^2 w \,, \tag{3.34}$$

where the symbol ∇^2 denotes the Laplace operator, $\nabla^2 = \partial^2/\partial x^2 + \partial^2/\partial y^2 + \partial^2/\partial z^2$. The above Navier-Stokes equations differ from Euler's equations of motion by the viscous terms $\mu \nabla^2 w$.

The solutions of the above equations become fully determined physically when the boundary and initial conditions are specified. In the case of viscous fluids the condition of no slip on solid boundaries must be satisfied, i. e., on a wall both the normal and tangential components of the velocity must vanish:

$$v_n = 0 \,, \quad v_t = 0 \text{ on solid walls} \,. \tag{3.35}$$

The equations under discussion were first derived by M. Navier [9] in 1827 and by S. D. Poisson [10] in 1831, on the basis of an argument which involved the consideration of intermolecular forces. Later the same equations were derived without the use of any such hypotheses by B. de Saint Venant [14] in 1843 and by G. G. Stokes [13] in 1845. Their derivations were based on the same assumption as made here, namely that the normal and shearing stresses are linear functions of the rate of deformation, in conformity with the older law of friction, due to Newton, and that the thermodynamic pressure is equal to one-third of the sum of the normal stresses taken with an opposite sign.

Since the hypothesis of linearity is evidently completely arbitrary, it is not a priori certain that the Navier-Stokes equations give a true description of the motion of a fluid. It is, therefore, necessary to verify them, and that can only be achieved by experiment. In this connexion it should, in any case, be noted that the enormous mathematical difficulties encountered when solving the Navier-Stokes equations have so far prevented us from obtaining a single analytic solution in which the convective terms interact in a general way with the friction terms. However, known solutions, such as laminar flow through a circular pipe, as well as boundary-layer flows, to be discussed later, agree so well with experiment that the general validity of the Navier-Stokes equations can hardly be doubted.

Cylindrical coordinates: We shall now transform the Navier-Stokes equations to cylindrical coordinates for future reference. If r, ϕ, z denote the radial, azimuthal, and axial coordinates, respectively, of a three-dimensional system of coordinates, and v_r, v_ϕ, v_z denote the velocity components in the respective directions, then the transformation of variables [3, 11] for the case of incompressible fluid flow, eqns. (3.33) and (3.34), leads to the following system of equations:

$$\varrho\left(\frac{\partial v_r}{\partial t} + v_r\frac{\partial v_r}{\partial r} + \frac{v_\phi}{r}\frac{\partial v_r}{\partial \phi} - \frac{v_\phi^2}{r} + v_z\frac{\partial v_r}{\partial z}\right) =$$

$$= F_r - \frac{\partial p}{\partial r} + \mu\left(\frac{\partial^2 v_r}{\partial r^2} + \frac{1}{r}\frac{\partial v_r}{\partial r} - \frac{v_r}{r^2} + \frac{1}{r^2}\frac{\partial^2 v_r}{\partial \phi^2} - \frac{2}{r^2}\frac{\partial v_\phi}{\partial \phi} + \frac{\partial^2 v_r}{\partial z^2}\right) \tag{3.36a}$$

$$\varrho\left(\frac{\partial v_\phi}{\partial t} + v_r\frac{\partial v_\phi}{\partial r} + \frac{v_\phi}{r}\frac{\partial v_\phi}{\partial \phi} + \frac{v_r v_\phi}{r} + v_z\frac{\partial v_\phi}{\partial z}\right) =$$

$$= F_\phi - \frac{1}{r}\frac{\partial p}{\partial \phi} + \mu\left(\frac{\partial^2 v_\phi}{\partial r^2} + \frac{1}{r}\frac{\partial v_\phi}{\partial r} - \frac{v_\phi}{r^2} + \frac{1}{r^2}\frac{\partial^2 v_\phi}{\partial \phi^2} + \frac{2}{r^2}\frac{\partial v_r}{\partial \phi} + \frac{\partial^2 v_\phi}{\partial z^2}\right) \tag{3.36b}$$

$$\varrho\left(\frac{\partial v_z}{\partial t} + v_r\frac{\partial v_z}{\partial r} + \frac{v_\phi}{r}\frac{\partial v_z}{\partial \phi} + v_z\frac{\partial v_z}{\partial z}\right) =$$

$$= F_z - \frac{\partial p}{\partial z} + \mu\left(\frac{\partial^2 v_z}{\partial r^2} + \frac{1}{r}\frac{\partial v_z}{\partial r} + \frac{1}{r^2}\frac{\partial^2 v_z}{\partial \phi^2} + \frac{\partial^2 v_z}{\partial z^2}\right) \tag{3.36c}$$

$$\frac{\partial v_r}{\partial r} + \frac{v_r}{r} + \frac{1}{r}\frac{\partial v_\phi}{\partial \phi} + \frac{\partial v_z}{\partial z} = 0 \,. \tag{3.36d}$$

The stress components assume the form

$$\left.\begin{aligned}
\sigma_r &= -p + 2\mu\frac{\partial v_r}{\partial r}\,; & \tau_{r\phi} &= \mu\left[r\frac{\partial}{\partial r}\left(\frac{v_\phi}{r}\right) + \frac{1}{r}\frac{\partial v_r}{\partial \phi}\right]\\
\sigma_\phi &= -p + 2\mu\left(\frac{1}{r}\frac{\partial v_\phi}{\partial \phi} + \frac{v_r}{r}\right)\,; & \tau_{\phi z} &= \mu\left(\frac{\partial v_\phi}{\partial z} + \frac{1}{r}\frac{\partial v_z}{\partial \phi}\right)\\
\sigma_z &= -p + 2\mu\frac{\partial v_z}{\partial z}\,; & \tau_{rz} &= \mu\left(\frac{\partial v_r}{\partial z} + \frac{\partial v_z}{\partial r}\right).
\end{aligned}\right\} \tag{3.37}$$

Curvilinear coordinates: It is often useful to employ a curvilinear system of coordinates which is adapted to the shape of the body. In the case of two-dimensional flow along a curved wall, it is possible to select a coordinate system whose abscissa, x, is measured along the wall, the ordinate, y, being measured at right angles to it, Fig. 3.9. Thus the curvilinear net consists of curves which are parallel to the wall

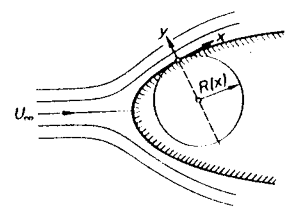

Fig. 3.9. Two-dimensional boundary layer along a curved wall

and of straight lines perpendicular to them. The corresponding velocity components are denoted by u and v, respectively. The radius of curvature at position x is denoted by $R(x)$; it is positive for walls which are convex outwards, and negative when the wall is concave. The appropriate form of the complete Navier-Stokes equations has been derived by W. Tollmien [15]. They are:

$$\begin{aligned}\frac{\partial u}{\partial t} + \frac{R}{R+y}\, u\, \frac{\partial u}{\partial x} + v\, \frac{\partial u}{\partial y} + \frac{v\, u}{R+y} = &- \frac{R}{R+y}\, \frac{1}{\varrho}\, \frac{\partial p}{\partial x} + \\ &+ \nu \left\{ \frac{R^2}{(R+y)^2}\, \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{1}{R+y}\, \frac{\partial u}{\partial y} - \frac{u}{(R+y)^2} + \right. \\ &\left. + \frac{2\,R}{(R+y)^2}\, \frac{\partial v}{\partial x} - \frac{R}{(R+y)^3}\, \frac{\mathrm{d}R}{\mathrm{d}x}\, v + \frac{R\,y}{(R+y)^3}\, \frac{\mathrm{d}R}{\mathrm{d}x}\, \frac{\partial u}{\partial x} \right\};\end{aligned} \qquad (3.38\,\mathrm{a})$$

$$\begin{aligned}\frac{\partial v}{\partial t} + \frac{R}{R+y}\, u\, \frac{\partial v}{\partial x} + v\, \frac{\partial v}{\partial y} - \frac{u^2}{R+y} = - \frac{1}{\varrho}\, \frac{\partial p}{\partial y} + \nu \left\{ \frac{\partial^2 v}{\partial y^2} - \frac{2\,R}{(R+y)^2}\, \frac{\partial u}{\partial x} + \right. \\ + \frac{1}{R+y}\, \frac{\partial v}{\partial y} + \frac{R^2}{(R+y)^2}\, \frac{\partial^2 v}{\partial x^2} - \frac{v}{(R+y)^2} + \\ \left. + \frac{R}{(R+y)^3}\, \frac{\mathrm{d}R}{\mathrm{d}x}\, u + \frac{R\,y}{(R+y)^3}\, \frac{\mathrm{d}R}{\mathrm{d}x}\, \frac{\partial v}{\partial x} \right\};\end{aligned} \qquad (3.38\,\mathrm{b})$$

$$\frac{R}{R+y}\, \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{v}{R+y} = 0\,. \qquad (3.38\,\mathrm{c})$$

The stress components are

$$\left.\begin{aligned}\sigma_x &= -p + 2\,\mu \left(\frac{R}{R+y}\, \frac{\partial u}{\partial x} + \frac{v}{R+y} \right) \\ \sigma_y &= -p + 2\,\mu\, \frac{\partial v}{\partial y} \\ \tau_{xy} &= \mu \left(\frac{\partial u}{\partial y} - \frac{u}{R+y} + \frac{R}{R+y}\, \frac{\partial v}{\partial x} \right),\end{aligned}\right\} \qquad (3.39)$$

and the vorticity [see eqn. (4.5)] becomes

$$\omega = \frac{1}{2} \left(\frac{R}{R+y}\, \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} - \frac{1}{R+y)}\, u \right). \qquad (3.40)$$

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