Viscous Flow in Channels: the continuity equation

In this chapter we shall deal with realistic situations in (x, y), where a liquid locally is at rest with respect to the solid objects in contact with it. Under such conditions curl($\mathbf{v}$) will in general be non-zero.

Classical mechanics applied to a liquid yields the Navier-Stokes equation. That equation expresses Newton's law of motion

$$\rho_0 \frac{d\mathbf{v}}{dt} = \mathbf{f}_{tot}$$

for the total force $\mathbf{f}_{tot}$ on a fluid element that is carried along with the stream. (That kind of derivative is also commonly denoted $\mathrm{D}\mathbf{v}/\mathrm{D}t$.) Here, ρ_0 is the *constant* mass density of the fluid. Since the velocity in a chosen volume element is a function of (t, x, y), we may write

$$\frac{d\mathbf{v}}{dt} = \frac{\partial \mathbf{v}}{\partial t} + \frac{\partial \mathbf{v}}{\partial x}\frac{dx}{dt} + \frac{\partial \mathbf{v}}{\partial y}\frac{dy}{dt} = \frac{\partial \mathbf{v}}{\partial t} + v_x \frac{\partial \mathbf{v}}{\partial x} + v_y \frac{\partial \mathbf{v}}{\partial y} = \frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{v}$$

With this expression for the derivative, Newton's law takes the form

$$\rho_0 \frac{\partial \mathbf{v}}{\partial t} + \rho_0 (\mathbf{v} \cdot \nabla)\mathbf{v} - \mathbf{F} + \nabla p - \eta \nabla^2 \mathbf{v} = 0 \qquad \bullet$$

where $\mathbf{F}$ is an *external* force (e.g. gravity), $-\nabla p$ the force due to pressure, and $\eta \nabla^2 \mathbf{v}$ the one proportional to viscosity. This vector PDE is known as the *Navier-Stokes* (N-S) equation.

The second term, $\rho_0 (\mathbf{v} \cdot \nabla)\mathbf{v}$, has the dimension of force but it is really part of the time derivative and hence called an *inertial* force. This term is obviously second-order in $\mathbf{v}$.

The last term corresponds to the *viscous* force on the volume element. Normally, ∇^2 operates on a scalar and $\nabla^2 \mathbf{v}$ should be taken as shorthand for the vector $(\mathbf{i}\nabla^2 v_x + \mathbf{j}\nabla^2 v_y)$.

The simplest case of flow occurs at such small speeds that the non-linear inertial force become negligible compared to viscous force, and here we shall consider liquid motion under such conditions. The ratio of inertial-to-viscous forces is usually expressed in the form of the dimensionless *Reynolds number*, defined by

$$\mathrm{Re} = \frac{\rho_0 v_0 L_0}{\eta} \qquad \bullet$$

where v_0 is a typical speed and L_0 a typical size of the solution domain. This number gives us an order-of-magnitude indication of the sort of flow we are dealing with. At sufficiently small values of Re, the inertial term is negligible compared to the viscous force and the problem can be treated as linear in the dependent variables. The PDEs then yield solutions corresponding to laminar flow.

Above the first critical value ($\mathrm{Re} = 1$) the solutions may remain laminar, even if the PDEs are non-linear. Above a much higher value (Re=100 or much more depending of the details of the problem) the solution becomes turbulent and time-dependent (permanently unstable).

In Cartesian coordinates, the component Navier-Stokes equations may thus be written (for the *x*- and *y*-directions respectively)

$$\rho_0 \begin{Bmatrix} \dfrac{\partial v_x}{\partial t} \\ \dfrac{\partial v_y}{\partial t} \end{Bmatrix} + \rho_0 (\mathbf{v} \cdot \nabla) \mathbf{v} - \begin{Bmatrix} F_x \\ F_y \end{Bmatrix} + \begin{Bmatrix} \dfrac{\partial p}{\partial x} \\ \dfrac{\partial p}{\partial y} \end{Bmatrix} - \eta \begin{Bmatrix} \dfrac{\partial^2 v_x}{\partial x^2} + \dfrac{\partial^2 v_x}{\partial y^2} \\ \dfrac{\partial^2 v_y}{\partial x^2} + \dfrac{\partial^2 v_y}{\partial y^2} \end{Bmatrix} = 0 \qquad \bullet$$

Here, we have kept the second term unexpanded, since it may be disregarded until a later chapter.

So far, we have only two equations for the three dependent variables v_x, v_y, and *p*. Conservation of mass at constant density gives us a third equation, i.e.

$$\nabla \cdot (\rho_0 \mathbf{v}) = \rho_0 \nabla \cdot \mathbf{v} = \rho_0 \left(\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} \right) = 0 \qquad \bullet$$

but unfortunately this is a PDE of first order only, which FlexPDE would not accept.

Using $\nabla \cdot \mathbf{v} = 0$ together with the equation of motion we may, however, generate a relation containing second-order derivatives in *p*. Applying the divergence operator to the N-S equation we obtain

$$\rho_0 \frac{\partial}{\partial t} \nabla \cdot \mathbf{v} + \rho_0 \nabla \cdot \left[(\mathbf{v} \cdot \nabla) \mathbf{v} \right] - \nabla \cdot \mathbf{F} + \nabla^2 p - \eta \nabla \cdot (\nabla^2 \mathbf{v}) = 0$$

where the first term vanishes because of mass conservation. Furthermore, we may eliminate the last term using the identities

$$\eta \nabla \cdot (\nabla^2 \mathbf{v}) = \eta \nabla^2 (\nabla \cdot \mathbf{v}) = \eta \nabla^2 (0) = 0$$

The remainder of the modified N-S equation is

$$\nabla^2 p + \rho_0 \nabla \cdot \left[(\mathbf{v} \cdot \nabla) \mathbf{v} \right] - \nabla \cdot \mathbf{F} = 0 \qquad \bullet$$

If the volume force **F** is constant in space the last term will vanish.

Expressed in Cartesian coordinates, this PDE takes the form

$$\frac{\partial^2 p}{\partial x^2} + \frac{\partial^2 p}{\partial y^2} + \rho_0 \nabla \cdot \left[(\mathbf{v} \cdot \nabla) \mathbf{v} \right] - \nabla \cdot \mathbf{F} = 0$$

Even in this equation we leave the term containing ρ_0 unexpanded, since it will not be used in the present chapter.

We now have a total of three PDEs for calculating v_x, v_y and *p*. Although we derived the equation for *p* using mass conservation, it would be wrong to assume that any solution to these three PDEs would necessarily satisfy $\nabla \cdot \mathbf{v}$=0. In fact, one may show that this is true only in special cases. We shall see that the first two examples in this chapter are sufficiently simple for the divergence to vanish automatically.

It could never be wrong, however, to add $\nabla \cdot \mathbf{v}$, multiplied by a factor, to the equation for *p*, since the divergence should vanish in the final stage of the solution process. Hence we settle for the following form

$$\frac{\partial^2 p}{\partial x^2} + \frac{\partial^2 p}{\partial y^2} + \rho_0 \nabla \cdot \left[(\mathbf{v} \cdot \nabla) \mathbf{v} \right] - \nabla \cdot \mathbf{F} - f_\nabla \nabla \cdot \mathbf{v} = 0 \qquad \bullet$$

where we may choose the factor f_∇ freely according to the problem at hand, to ensure vanishing divergence. Trial runs lead us to employ a negative factor. We may always verify by means of plots that the divergence vanishes for a given solution.

The factor f_∇ may not be taken as a fixed number, however, since it has a physical dimension, in fact the same as η / L_0^2. Hence, we should write

$$f_\nabla = C\frac{\eta}{L_0^2}$$ ●

where the parameter L_0 is a typical size of the domain. The number C is to be chosen empirically, large enough to ensure vanishing $\nabla \cdot \mathbf{v}$, but not so large that it impairs convergence in FlexPDE calculations or requires unreasonably long runtimes.

Although the divergence term was introduced on intuitive grounds and proves itself in practical use, we may understand approximately how it works. ● If the term $\mathbf{f} = -\nabla p$ is the force generated by pressure, the Gauss theorem yields

$$\iiint \nabla^2 p\, dV = \iiint \nabla \cdot \nabla p\, dV = -\iiint \nabla \cdot \mathbf{f}\, dV = -\oiint f_n\, ds$$

Let us now consider a small region around a point of interest. By subtracting a certain amount from the $\nabla^2 p$ term in page3● we effectively create an outward force on the boundary of that region, which transports fluid away from the point considered. This nudges the calculations toward vanishing divergence.

Boundary Conditions

Now that we have a PDE for pressure, we must find out what boundary conditions to use with it. This is easy enough where the pressure takes known values, but what about boundaries that just limit the fluid flow?

The alternative to *value* is a *natural* statement. In the latter case we need an expression for $\partial p / \partial n \equiv \mathbf{n} \cdot \nabla p$, where $\mathbf{n}$ is the outward normal ($|\mathbf{n}| = 1$) at the boundary of the domain. The N-S equation (page 1) provides the answer rather directly :

$$\nabla p = \mathbf{F} + \eta \nabla^2 \mathbf{v} - \rho_0 \frac{\partial \mathbf{v}}{\partial t} - \rho_0 (\mathbf{v} \cdot \nabla)\, \mathbf{v}$$

If the pressure is not known on a boundary segment, we may thus use the following general expression for the *natural* boundary condition

$$\partial p / \partial n \;= \mathbf{n} \cdot \nabla p = \mathbf{n} \cdot \mathbf{F} + \eta \mathbf{n} \cdot \nabla^2 \mathbf{v} - \rho_0 \mathbf{n} \cdot \frac{\partial \mathbf{v}}{\partial t} - \rho_0\, \mathbf{n} \cdot [(\mathbf{v} \cdot \nabla)\, \mathbf{v}] =$$

$$n_x F_x + n_y F_y + \eta \left[n_x \nabla^2 v_x + n_y\, \nabla^2 v_y \right] - \rho_0 \mathbf{n} \cdot \frac{\partial \mathbf{v}}{\partial t} - \rho_0\, \mathbf{n} \cdot [(\mathbf{v} \cdot \nabla)\, \mathbf{v}] \qquad \bullet$$

where $\rho_0\, \partial v_n / \partial t$ will vanish in the steady state, and we defer the expansion of the last term until it is required later.

Steady Flow at Small Speeds (Re<<1)

In this chapter and the next one we shall only be concerned with steady flow, which means that we omit the time derivative. We also assume Re to be small enough to permit us to neglect the PDE term proportional to the density. The three PDEs then take the simpler form

$$\begin{Bmatrix} \dfrac{\partial p}{\partial x} \\ \dfrac{\partial p}{\partial y} \end{Bmatrix} - \eta \begin{Bmatrix} \dfrac{\partial^2 v_x}{\partial x^2} + \dfrac{\partial^2 v_x}{\partial y^2} \\ \dfrac{\partial^2 v_y}{\partial x^2} + \dfrac{\partial^2 v_y}{\partial y^2} \end{Bmatrix} = 0 \qquad \bullet$$

$$\frac{\partial^2 p}{\partial x^2} + \frac{\partial^2 p}{\partial y^2} - C \frac{\eta}{L_0^2} \left(\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} \right) = 0 \qquad \bullet$$

We shall soon see that in the most elementary examples, involving parallel flow, we may even neglect the last (divergence) term.

For small Re, the *natural* boundary condition for pressure simplifies into

$$\partial p / \partial n \;= n_x F_x + n_y F_y + \eta \left(n_x \nabla^2 v_x + n_y \nabla^2 v_y \right) \qquad \bullet$$