CASO A

$$\frac{\partial \phi}{\partial t} = \kappa \frac{\partial^2 \phi}{\partial u^2} \quad \text{(a.1)}$$

con le condizioni al contorno

$$\begin{cases} \phi(u,0) = \phi_o \\ \phi(0,t) = \phi_w \\ \phi(\infty,t) = \phi_o \end{cases} \quad \text{(a.2)}$$

La soluzione è $\dfrac{\phi - \phi_w}{\phi_o - \phi_w} = erf(\eta)$ (a.3) con $\eta = \dfrac{u}{\sqrt{4\kappa t}}$ (a.4)

CASO B

$$\frac{\partial\phi}{\partial t}=\kappa\frac{\partial^2\phi}{\partial u^2} \quad \text{(b.1)}$$ con le condizioni al contorno $$\begin{cases}\phi(u,0)=0 \qquad per \;\; u>0\\ \dfrac{\partial\phi}{\partial u}(0,t)=0\\ \phi(\infty,t)=0\\ M=\displaystyle\int_{u=0}^{\infty}\phi(u,t)\;\;du=\text{costante}\end{cases} \quad \text{(b.2)}$$

Introduciamo una funzione ψ tale da soddisfare l'equazione

$$\frac{\partial\psi}{\partial t}=\kappa\frac{\partial^2\psi}{\partial u^2} \quad \text{(b.3)}$$ con le condizioni al contorno $$\begin{cases}\psi(u,0)=\psi_o\\ \psi(0,t)=\psi_w\\ \psi(\infty,t)=\psi_o\end{cases} \quad \text{(b.4)}$$

Riferendoci al caso A, ψ ha espressione

$$\frac{\psi-\psi_w}{\psi_o-\psi_w}=erf(\eta) \quad \text{(b.5)} \qquad \text{con} \qquad \eta=\frac{u}{\sqrt{4\kappa t}} \quad \text{(b.6)}$$

Sia ora $\phi=\dfrac{\partial\psi}{\partial u}$ (b.7).

Anche ϕ soddisfa l'eq. differenziale $\dfrac{\partial\phi}{\partial t}=\kappa\dfrac{\partial^2\phi}{\partial u^2}$ (b.8) con le condizioni al contorno $\begin{cases}\phi(u,0)=0\\ \dfrac{\partial\phi}{\partial u}(0,t)=0\\ \phi(\infty,t)=0\end{cases}$ (b.9)

L'espressione di ϕ si ricava derivando l'eq. (b.5):

$$\phi=\frac{\partial\psi}{\partial u}=(\psi_o-\psi_w)\frac{\partial erf(\eta)}{\partial\eta}\frac{\partial\eta}{\partial u}=(\psi_o-\psi_w)\frac{e^{-\left(\frac{u}{\sqrt{4\kappa t}}\right)^2}}{\sqrt{\pi\kappa t}}$$

La costante (ψ_o-ψ_w) si ottiene imponendo $M=\displaystyle\int_{u=0}^{\infty}\phi(u,t)\;\;du$, ossia

$$M=\int_{u=0}^{\infty}\frac{\partial\psi}{\partial u}\;\;du=\psi_o-\psi_w$$

In definitiva la soluzione è

$$\phi=M\frac{e^{-\left(\frac{u}{\sqrt{4\kappa t}}\right)^2}}{\sqrt{\pi\kappa t}}$$

CASO C

$$\frac{\partial\phi}{\partial t}=\kappa\frac{\partial^2\phi}{\partial u^2} \quad \text{(c.1)}$$

con le condizioni al contorno

$$\begin{cases}\phi(u,0)=\phi_o\\ -\gamma\dfrac{\partial\phi}{\partial u}(0,t)=\theta_w \quad \text{(c.2)}\\ \phi(\infty,t)=\phi_o\end{cases}$$

Introduciamo $\psi=\dfrac{\partial\phi}{\partial u}$ (c.3). Anche ψ soddisfa l'equazione

$$\frac{\partial\psi}{\partial t}=\kappa\frac{\partial^2\psi}{\partial u^2} \quad \text{(c.4)}$$

con le condizioni al contorno

$$\begin{cases}\psi(u,0)=0\\ \psi(0,t)=\psi_w=-\theta_w/\gamma \quad \text{(c.5)}\\ \psi(\infty,t)=0\end{cases}$$

Riferendoci al caso A, ψ ha espressione

$$\frac{\psi-\psi_w}{-\psi_w}=erf(\eta) \quad \text{(c.6)} \qquad \text{con} \qquad \eta=\frac{u}{\sqrt{4\kappa t}} \quad \text{(c.7)}$$

L'espressione di ϕ si ricava integrando l'eq. (c.6):

$$\phi=\psi_w\int\left(1-erf(\eta)\right)du+c=\ \psi_w\int\frac{\left(1-erf(\eta)\right)}{\frac{\partial\eta}{\partial u}}d\eta+c=\ \psi_w\sqrt{4\kappa t}\int\left(1-erf(\eta)\right)d\eta+c=$$

$$=\psi_w\sqrt{4\kappa t}\left[\eta\left(1-erf(\eta)\right)-\frac{e^{-\eta^2}}{\sqrt{\pi}}\right]+c\quad=\phi_o+\frac{\theta_w}{\gamma}\sqrt{4\kappa t}\left[\eta\left(1-erf(\eta)\right)-\frac{e^{-\eta^2}}{\sqrt{\pi}}\right]$$

Oss.: $\int erf(\eta)d\eta=\ \eta erf(\eta)-\int\frac{2}{\sqrt{\pi}}\eta e^{-\eta^2}d\eta=\eta erf(\eta)+\int\frac{1}{\sqrt{\pi}}e^x dx\Big|_{x=-\eta^2}=\eta erf(\eta)+\frac{e^{-\eta^2}}{\sqrt{\pi}}$

Oss.: $\lim_{\eta\to\infty}\psi_w\sqrt{4\kappa t}\left[\eta\left(1-erf(\eta)\right)-\frac{e^{-\eta^2}}{\sqrt{\pi}}\right]+c=c$

CASO D

$$\frac{\partial\phi}{\partial t}=\kappa\frac{\partial^2\phi}{\partial u^2} \quad \text{(d.1)} \qquad \text{con le condizioni al contorno} \qquad \begin{cases}\phi(u,0)=\phi_o\\ \gamma\frac{\partial\phi}{\partial u}(0,t)=\varepsilon\big(\phi(0,t)-\phi_e\big) \text{(d.2)}\\ \phi(\infty,t)=\phi_o\end{cases}$$

Introduciamo $\tilde{\phi}=\frac{\phi-\phi_e}{\phi_o-\phi_e}$ (d.3). Otteniamo

$$\frac{\partial\tilde{\phi}}{\partial t}=\kappa\frac{\partial^2\tilde{\phi}}{\partial u^2} \quad \text{(d.4)} \qquad \text{con le condizioni al contorno} \qquad \begin{cases}\tilde{\phi}(u,0)=1\\ \frac{\partial\tilde{\phi}}{\partial u}(0,t)-\frac{\varepsilon}{\gamma}\phi(0,t)=0\\ \tilde{\phi}(\infty,t)=1\end{cases} \quad \text{(d.5)}$$

Introduciamo una funzione $\psi=\tilde{\phi}-\frac{\gamma}{\varepsilon}\frac{\partial\tilde{\phi}}{\partial u}$ (d.6). Otteniamo

$$\frac{\partial\psi}{\partial t}=\kappa\frac{\partial^2\psi}{\partial u^2} \quad \text{(d.7)} \qquad \text{con le condizioni al contorno} \qquad \begin{cases}\psi(u,0)=1\\ \psi(0,t)=0 \quad \text{(d.8)}\\ \psi(\infty,t)=1\end{cases}$$

Riferendoci al caso A, ψ ha espressione

$\psi=erf(\eta)$ (d.9) con $\eta=\frac{u}{\sqrt{4\kappa t}}$ (d.10)

L'eq. (d.6) si risolve moltiplicando tutto per un fattore integrante $e^{-\frac{\varepsilon}{\gamma}u}$:

$$-\frac{\varepsilon}{\gamma}\psi e^{-\frac{\varepsilon}{\gamma}u}=-\frac{\varepsilon}{\gamma}\tilde{\phi}e^{-\frac{\varepsilon}{\gamma}u}+\frac{\partial\tilde{\phi}}{\partial u}e^{-\frac{\varepsilon}{\gamma}u}=\frac{\partial\tilde{\phi}e^{-\frac{\varepsilon}{\gamma}u}}{\partial u}\Rightarrow\frac{\partial\tilde{\phi}e^{-\frac{\varepsilon}{\gamma}u}}{\partial u}=-\frac{\varepsilon}{\gamma}erf\left(\frac{u}{\sqrt{4\kappa t}}\right)e^{-\frac{\varepsilon}{\gamma}u} \quad \text{(d.11)}$$

$$\tilde{\phi}=-\frac{\varepsilon}{\gamma}e^{\frac{\varepsilon}{\gamma}u}\int_\infty^u erf\left(\frac{u}{\sqrt{4\kappa t}}\right)e^{-\frac{\varepsilon}{\gamma}u}=erf\left(\frac{u}{\sqrt{4\kappa t}}\right)+e^{\left(\frac{\varepsilon}{\gamma}\sqrt{\kappa t}\right)^2+\frac{\varepsilon}{\gamma}u}\left[1-erf\left(\frac{u}{\sqrt{4\kappa t}}+\frac{\varepsilon}{\gamma}\sqrt{\kappa t}\right)\right] \quad \text{(d.12)}$$

Oss.:

$$\int_\infty^u erf\left(\frac{u}{\sqrt{4\kappa t}}\right)e^{-\frac{\varepsilon}{\gamma}u}du=-\frac{\gamma}{\varepsilon}e^{-\frac{\varepsilon}{\gamma}u}erf\left(\frac{u}{\sqrt{4\kappa t}}\right)\Bigg|_\infty^u+\frac{\gamma}{\varepsilon}\frac{1}{\sqrt{\pi\kappa t}}\int_\infty^u e^{-\frac{u^2}{4\kappa t}}e^{-\frac{\varepsilon}{\gamma}u}du=-\frac{\gamma}{\varepsilon}e^{-\frac{\varepsilon}{\gamma}u}erf\left(\frac{u}{\sqrt{4\kappa t}}\right)+\frac{\gamma}{\varepsilon}\frac{1}{\sqrt{\pi\kappa t}}\int_\infty^u e^{-\left(\frac{u^2}{4\kappa t}+\frac{\varepsilon}{\gamma}u\right)}du=$$

$$=-\frac{\gamma}{\varepsilon}e^{-\frac{\varepsilon}{\gamma}u}erf\left(\frac{u}{\sqrt{4\kappa t}}\right)+\frac{\gamma}{\varepsilon}\frac{1}{\sqrt{\pi\kappa t}}\int_\infty^u e^{-\left(\frac{u}{\sqrt{4\kappa t}}+\frac{\varepsilon}{2\gamma}\sqrt{4\kappa t}\right)^2}e^{\left(\frac{\varepsilon}{2\gamma}\sqrt{4\kappa t}\right)^2}du=-\frac{\gamma}{\varepsilon}e^{-\frac{\varepsilon}{\gamma}u}erf\left(\frac{u}{\sqrt{4\kappa t}}\right)+\frac{\gamma}{\varepsilon}\frac{e^{\left(\frac{\varepsilon}{2\gamma}\sqrt{4\kappa t}\right)^2}}{\sqrt{\pi\kappa t}}\sqrt{4\kappa t}\int_\infty^{x=\frac{u}{\sqrt{4\kappa t}}+\frac{\varepsilon}{2\gamma}\sqrt{4\kappa t}}e^{-x^2}\,dx=$$

$$=-\frac{\gamma}{\varepsilon}e^{-\frac{\varepsilon}{\gamma}u}erf\left(\frac{u}{\sqrt{4\kappa t}}\right)+\frac{\gamma}{\varepsilon}\frac{e^{\left(\frac{\varepsilon}{2\gamma}\sqrt{4\kappa t}\right)^2}}{\sqrt{\pi}}2\left[\int_\infty^0 e^{-x^2}dx+\int_0^{x=\frac{u}{\sqrt{4\kappa t}}+\frac{\varepsilon}{2\gamma}\sqrt{4\kappa t}}e^{-x^2}\,dx\right]-\frac{\gamma}{\varepsilon}e^{-\frac{\varepsilon}{\gamma}u}erf\left(\frac{u}{\sqrt{4\kappa t}}\right)-\frac{\gamma}{\varepsilon}e^{\left(\frac{\varepsilon}{2\gamma}\sqrt{4\kappa t}\right)^2}\left[1-erf\left(\frac{u}{\sqrt{4\kappa t}}+\frac{\varepsilon}{2\gamma}\sqrt{4\kappa t}\right)\right]$$