

The Fluxes and the Equations of Change

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§B.1 NEWTON'S LAW OF VISCOSITY

$$[\tau = -\mu(\nabla\mathbf{v} + (\nabla\mathbf{v})^\dagger) + (\frac{2}{3}\mu - \kappa)(\nabla \cdot \mathbf{v})\delta]$$

Cartesian coordinates (x, y, z):

$$\tau_{xx} = -\mu \left[2 \frac{\partial v_x}{\partial x} \right] + (\frac{2}{3}\mu - \kappa)(\nabla \cdot \mathbf{v}) \quad (\text{B.1-1})^a$$

$$\tau_{yy} = -\mu \left[2 \frac{\partial v_y}{\partial y} \right] + (\frac{2}{3}\mu - \kappa)(\nabla \cdot \mathbf{v}) \quad (\text{B.1-2})^a$$

$$\tau_{zz} = -\mu \left[2 \frac{\partial v_z}{\partial z} \right] + (\frac{2}{3}\mu - \kappa)(\nabla \cdot \mathbf{v}) \quad (\text{B.1-3})^a$$

$$\tau_{xy} = \tau_{yx} = -\mu \left[\frac{\partial v_y}{\partial x} + \frac{\partial v_x}{\partial y} \right] \quad (\text{B.1-4})$$

$$\tau_{yz} = \tau_{zy} = -\mu \left[\frac{\partial v_z}{\partial y} + \frac{\partial v_y}{\partial z} \right] \quad (\text{B.1-5})$$

$$\tau_{zx} = \tau_{xz} = -\mu \left[\frac{\partial v_x}{\partial z} + \frac{\partial v_z}{\partial x} \right] \quad (\text{B.1-6})$$

in which

$$(\nabla \cdot \mathbf{v}) = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z} \quad (\text{B.1-7})$$

^a When the fluid is assumed to have constant density, the term containing $(\nabla \cdot \mathbf{v})$ may be omitted. For monatomic gases at low density, the dilatational viscosity κ is zero.

§B.1 NEWTON'S LAW OF VISCOSITY (continued)

Cylindrical coordinates (r, θ, z):

$$\tau_{rr} = -\mu \left[2 \frac{\partial v_r}{\partial r} \right] + \left(\frac{2}{3}\mu - \kappa \right) (\nabla \cdot \mathbf{v}) \quad (\text{B.1-8})^a$$

$$\tau_{\theta\theta} = -\mu \left[2 \left(\frac{1}{r} \frac{\partial v_\theta}{\partial \theta} + \frac{v_r}{r} \right) \right] + \left(\frac{2}{3}\mu - \kappa \right) (\nabla \cdot \mathbf{v}) \quad (\text{B.1-9})^a$$

$$\tau_{zz} = -\mu \left[2 \frac{\partial v_z}{\partial z} \right] + \left(\frac{2}{3}\mu - \kappa \right) (\nabla \cdot \mathbf{v}) \quad (\text{B.1-10})^a$$

$$\tau_{r\theta} = \tau_{\theta r} = -\mu \left[r \frac{\partial}{\partial r} \left(\frac{v_\theta}{r} \right) + \frac{1}{r} \frac{\partial v_r}{\partial \theta} \right] \quad (\text{B.1-11})$$

$$\tau_{\theta z} = \tau_{z\theta} = -\mu \left[\frac{1}{r} \frac{\partial v_z}{\partial \theta} + \frac{\partial v_\theta}{\partial z} \right] \quad (\text{B.1-12})$$

$$\tau_{zr} = \tau_{rz} = -\mu \left[\frac{\partial v_r}{\partial z} + \frac{\partial v_z}{\partial r} \right] \quad (\text{B.1-13})$$

in which

$$(\nabla \cdot \mathbf{v}) = \frac{1}{r} \frac{\partial}{\partial r} (rv_r) + \frac{1}{r} \frac{\partial v_\theta}{\partial \theta} + \frac{\partial v_z}{\partial z} \quad (\text{B.1-14})$$

^a When the fluid is assumed to have constant density, the term containing $(\nabla \cdot \mathbf{v})$ may be omitted. For monatomic gases at low density, the dilatational viscosity κ is zero.

Spherical coordinates (r, θ, ϕ):

$$\tau_{rr} = -\mu \left[2 \frac{\partial v_r}{\partial r} \right] + \left(\frac{2}{3}\mu - \kappa \right) (\nabla \cdot \mathbf{v}) \quad (\text{B.1-15})^a$$

$$\tau_{\theta\theta} = -\mu \left[2 \left(\frac{1}{r} \frac{\partial v_\theta}{\partial \theta} + \frac{v_r}{r} \right) \right] + \left(\frac{2}{3}\mu - \kappa \right) (\nabla \cdot \mathbf{v}) \quad (\text{B.1-16})^a$$

$$\tau_{\phi\phi} = -\mu \left[2 \left(\frac{1}{r \sin \theta} \frac{\partial v_\phi}{\partial \phi} + \frac{v_r + v_\theta \cot \theta}{r} \right) \right] + \left(\frac{2}{3}\mu - \kappa \right) (\nabla \cdot \mathbf{v}) \quad (\text{B.1-17})^a$$

$$\tau_{r\theta} = \tau_{\theta r} = -\mu \left[r \frac{\partial}{\partial r} \left(\frac{v_\theta}{r} \right) + \frac{1}{r} \frac{\partial v_r}{\partial \theta} \right] \quad (\text{B.1-18})$$

$$\tau_{\theta\phi} = \tau_{\phi\theta} = -\mu \left[\frac{\sin \theta}{r} \frac{\partial}{\partial \theta} \left(\frac{v_\phi}{\sin \theta} \right) + \frac{1}{r \sin \theta} \frac{\partial v_\theta}{\partial \phi} \right] \quad (\text{B.1-19})$$

$$\tau_{\phi r} = \tau_{r\phi} = -\mu \left[\frac{1}{r \sin \theta} \frac{\partial v_r}{\partial \phi} + r \frac{\partial}{\partial r} \left(\frac{v_\phi}{r} \right) \right] \quad (\text{B.1-20})$$

in which

$$(\nabla \cdot \mathbf{v}) = \frac{1}{r^2} \frac{\partial}{\partial r} (r^2 v_r) + \frac{1}{r \sin \theta} \frac{\partial}{\partial \theta} (v_\theta \sin \theta) + \frac{1}{r \sin \theta} \frac{\partial v_\phi}{\partial \phi} \quad (\text{B.1-21})$$

^a When the fluid is assumed to have constant density, the term containing $(\nabla \cdot \mathbf{v})$ may be omitted. For monatomic gases at low density, the dilatational viscosity κ is zero.

§B.2 FOURIER'S LAW OF HEAT CONDUCTION^a

$$[\mathbf{q} = -k\nabla T]$$

Cartesian coordinates (x, y, z):

$$q_x = -k \frac{\partial T}{\partial x} \quad (\text{B.2-1})$$

$$q_y = -k \frac{\partial T}{\partial y} \quad (\text{B.2-2})$$

$$q_z = -k \frac{\partial T}{\partial z} \quad (\text{B.2-3})$$

Cylindrical coordinates (r, θ, z):

$$q_r = -k \frac{dT}{dr} \quad (\text{B.2-4})$$

$$q_\theta = -k \frac{1}{r} \frac{\partial T}{\partial \theta} \quad (\text{B.2-5})$$

$$q_z = -k \frac{\partial T}{\partial z} \quad (\text{B.2-6})$$

Spherical coordinates (r, θ, φ):

$$q_r = -k \frac{\partial T}{\partial r} \quad (\text{B.2-7})$$

$$q_\theta = -k \frac{1}{r} \frac{\partial T}{\partial \theta} \quad (\text{B.2-8})$$

$$q_\phi = -k \frac{1}{r \sin \theta} \frac{\partial T}{\partial \phi} \quad (\text{B.2-9})$$

^a For mixtures, the term $\sum_\alpha (\bar{H}_\alpha / M_\alpha) \mathbf{j}_\alpha$ must be added to \mathbf{q} (see Eq. 19.3-3).

§B.3 FICK'S (FIRST) LAW OF BINARY DIFFUSION^a

$$[\mathbf{j}_A = -\rho \mathcal{D}_{AB} \nabla \omega_A]$$

Cartesian coordinates (x, y, z):

$$j_{Ax} = -\rho \mathcal{D}_{AB} \frac{\partial \omega_A}{\partial x} \quad (\text{B.3-1})$$

$$j_{Ay} = -\rho \mathcal{D}_{AB} \frac{\partial \omega_A}{\partial y} \quad (\text{B.3-2})$$

$$j_{Az} = -\rho \mathcal{D}_{AB} \frac{\partial \omega_A}{\partial z} \quad (\text{B.3-3})$$

Cylindrical coordinates (r, θ, z):

$$j_{Ar} = -\rho \mathcal{D}_{AB} \frac{\partial \omega_A}{\partial r} \quad (\text{B.3-4})$$

$$j_{A\theta} = -\rho \mathcal{D}_{AB} \frac{1}{r} \frac{\partial \omega_A}{\partial \theta} \quad (\text{B.3-5})$$

$$j_{Az} = -\rho \mathcal{D}_{AB} \frac{\partial \omega_A}{\partial z} \quad (\text{B.3-6})$$

Spherical coordinates (r, θ, φ):

$$j_{Ar} = -\rho \mathcal{D}_{AB} \frac{\partial \omega_A}{\partial r} \quad (\text{B.3-7})$$

$$j_{A\theta} = -\rho \mathcal{D}_{AB} \frac{1}{r} \frac{\partial \omega_A}{\partial \theta} \quad (\text{B.3-8})$$

$$j_{A\phi} = -\rho \mathcal{D}_{AB} \frac{1}{r \sin \theta} \frac{\partial \omega_A}{\partial \phi} \quad (\text{B.3-9})$$

^a To get the molar fluxes with respect to the molar average velocity, replace \mathbf{j}_A , ρ , and ω_A by \mathbf{J}_A^* , c , and x_A .

§B.4 THE EQUATION OF CONTINUITY^a

$$[\partial \rho / \partial t + (\nabla \cdot \rho \mathbf{v}) = 0]$$

Cartesian coordinates (x, y, z):

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x} (\rho v_x) + \frac{\partial}{\partial y} (\rho v_y) + \frac{\partial}{\partial z} (\rho v_z) = 0 \quad (\text{B.4-1})$$

Cylindrical coordinates (r, θ, z):

$$\frac{\partial \rho}{\partial t} + \frac{1}{r} \frac{\partial}{\partial r} (\rho r v_r) + \frac{1}{r} \frac{\partial}{\partial \theta} (\rho v_\theta) + \frac{\partial}{\partial z} (\rho v_z) = 0 \quad (\text{B.4-2})$$

Spherical coordinates (r, θ, φ):

$$\frac{\partial \rho}{\partial t} + \frac{1}{r^2} \frac{\partial}{\partial r} (\rho r^2 v_r) + \frac{1}{r \sin \theta} \frac{\partial}{\partial \theta} (\rho v_\theta \sin \theta) + \frac{1}{r \sin \theta} \frac{\partial}{\partial \phi} (\rho v_\phi) = 0 \quad (\text{B.4-3})$$

^a When the fluid is assumed to have constant mass density ρ , the equation simplifies to $(\nabla \cdot \mathbf{v}) = 0$.

§B.5 THE EQUATION OF MOTION IN TERMS OF τ

$$[\rho D\mathbf{v}/Dt = -\nabla p - [\nabla \cdot \boldsymbol{\tau}] + \rho \mathbf{g}]$$

Cartesian coordinates (x, y, z):^a

$$\rho \left(\frac{\partial v_x}{\partial t} + v_x \frac{\partial v_x}{\partial x} + v_y \frac{\partial v_x}{\partial y} + v_z \frac{\partial v_x}{\partial z} \right) = -\frac{\partial p}{\partial x} - \left[\frac{\partial}{\partial x} \tau_{xx} + \frac{\partial}{\partial y} \tau_{yx} + \frac{\partial}{\partial z} \tau_{zx} \right] + \rho g_x \quad (\text{B.5-1})$$

$$\rho \left(\frac{\partial v_y}{\partial t} + v_x \frac{\partial v_y}{\partial x} + v_y \frac{\partial v_y}{\partial y} + v_z \frac{\partial v_y}{\partial z} \right) = -\frac{\partial p}{\partial y} - \left[\frac{\partial}{\partial x} \tau_{xy} + \frac{\partial}{\partial y} \tau_{yy} + \frac{\partial}{\partial z} \tau_{zy} \right] + \rho g_y \quad (\text{B.5-2})$$

$$\rho \left(\frac{\partial v_z}{\partial t} + v_x \frac{\partial v_z}{\partial x} + v_y \frac{\partial v_z}{\partial y} + v_z \frac{\partial v_z}{\partial z} \right) = -\frac{\partial p}{\partial z} - \left[\frac{\partial}{\partial x} \tau_{xz} + \frac{\partial}{\partial y} \tau_{yz} + \frac{\partial}{\partial z} \tau_{zz} \right] + \rho g_z \quad (\text{B.5-3})$$

^a These equations have been written without making the assumption that $\boldsymbol{\tau}$ is symmetric. This means, for example, that when the usual assumption is made that the stress tensor is symmetric, τ_{xy} and τ_{yx} may be interchanged.

Cylindrical coordinates (r, θ , z):^b

$$\rho \left(\frac{\partial v_r}{\partial t} + v_r \frac{\partial v_r}{\partial r} + \frac{v_\theta}{r} \frac{\partial v_r}{\partial \theta} + v_z \frac{\partial v_r}{\partial z} - \frac{v_\theta^2}{r} \right) = -\frac{\partial p}{\partial r} - \left[\frac{1}{r} \frac{\partial}{\partial r} (r\tau_{rr}) + \frac{1}{r} \frac{\partial}{\partial \theta} \tau_{\theta r} + \frac{\partial}{\partial z} \tau_{zr} - \frac{\tau_{\theta\theta}}{r} \right] + \rho g_r \quad (\text{B.5-4})$$

$$\rho \left(\frac{\partial v_\theta}{\partial t} + v_r \frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r} \frac{\partial v_\theta}{\partial \theta} + v_z \frac{\partial v_\theta}{\partial z} + \frac{v_r v_\theta}{r} \right) = -\frac{1}{r} \frac{\partial p}{\partial \theta} - \left[\frac{1}{r^2} \frac{\partial}{\partial r} (r^2 \tau_{r\theta}) + \frac{1}{r} \frac{\partial}{\partial \theta} \tau_{\theta\theta} + \frac{\partial}{\partial z} \tau_{z\theta} + \frac{\tau_{\theta r} - \tau_{r\theta}}{r} \right] + \rho g_\theta \quad (\text{B.5-5})$$

$$\rho \left(\frac{\partial v_z}{\partial t} + v_r \frac{\partial v_z}{\partial r} + \frac{v_\theta}{r} \frac{\partial v_z}{\partial \theta} + v_z \frac{\partial v_z}{\partial z} \right) = -\frac{\partial p}{\partial z} - \left[\frac{1}{r} \frac{\partial}{\partial r} (r\tau_{rz}) + \frac{1}{r} \frac{\partial}{\partial \theta} \tau_{\theta z} + \frac{\partial}{\partial z} \tau_{zz} \right] + \rho g_z \quad (\text{B.5-6})$$

^b These equations have been written without making the assumption that $\boldsymbol{\tau}$ is symmetric. This means, for example, that when the usual assumption is made that the stress tensor is symmetric, $\tau_{r\theta} - \tau_{\theta r} = 0$.

Spherical coordinates (r, θ , ϕ):^c

$$\rho \left(\frac{\partial v_r}{\partial t} + v_r \frac{\partial v_r}{\partial r} + \frac{v_\theta}{r} \frac{\partial v_r}{\partial \theta} + \frac{v_\phi}{r \sin \theta} \frac{\partial v_r}{\partial \phi} - \frac{v_\theta^2 + v_\phi^2}{r} \right) = -\frac{\partial p}{\partial r} - \left[\frac{1}{r^2} \frac{\partial}{\partial r} (r^2 \tau_{rr}) + \frac{1}{r \sin \theta} \frac{\partial}{\partial \theta} (\tau_{\theta r} \sin \theta) + \frac{1}{r \sin \theta} \frac{\partial}{\partial \phi} \tau_{\phi r} - \frac{\tau_{\theta\theta} + \tau_{\phi\phi}}{r} \right] + \rho g_r \quad (\text{B.5-7})$$

$$\rho \left(\frac{\partial v_\theta}{\partial t} + v_r \frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r} \frac{\partial v_\theta}{\partial \theta} + \frac{v_\phi}{r \sin \theta} \frac{\partial v_\theta}{\partial \phi} + \frac{v_r v_\theta - v_\phi^2 \cot \theta}{r} \right) = -\frac{1}{r} \frac{\partial p}{\partial \theta} - \left[\frac{1}{r^3} \frac{\partial}{\partial r} (r^3 \tau_{r\theta}) + \frac{1}{r \sin \theta} \frac{\partial}{\partial \theta} (\tau_{\theta\theta} \sin \theta) + \frac{1}{r \sin \theta} \frac{\partial}{\partial \phi} \tau_{\phi\theta} + \frac{(\tau_{\theta r} - \tau_{r\theta}) - \tau_{\phi\phi} \cot \theta}{r} \right] + \rho g_\theta \quad (\text{B.5-8})$$

$$\rho \left(\frac{\partial v_\phi}{\partial t} + v_r \frac{\partial v_\phi}{\partial r} + \frac{v_\theta}{r} \frac{\partial v_\phi}{\partial \theta} + \frac{v_\phi}{r \sin \theta} \frac{\partial v_\phi}{\partial \phi} + \frac{v_\phi v_r + v_\theta v_\phi \cot \theta}{r} \right) = -\frac{1}{r \sin \theta} \frac{\partial p}{\partial \phi} - \left[\frac{1}{r^3} \frac{\partial}{\partial r} (r^3 \tau_{r\phi}) + \frac{1}{r \sin \theta} \frac{\partial}{\partial \theta} (\tau_{\theta\phi} \sin \theta) + \frac{1}{r \sin \theta} \frac{\partial}{\partial \phi} \tau_{\phi\phi} + \frac{(\tau_{\phi r} - \tau_{r\phi}) + \tau_{\phi\theta} \cot \theta}{r} \right] + \rho g_\phi \quad (\text{B.5-9})$$

^c These equations have been written without making the assumption that $\boldsymbol{\tau}$ is symmetric. This means, for example, that when the usual assumption is made that the stress tensor is symmetric, $\tau_{r\theta} - \tau_{\theta r} = 0$.

§B.6 EQUATION OF MOTION FOR A NEWTONIAN FLUID WITH CONSTANT ρ AND μ

$$[\rho D\mathbf{v}/Dt = -\nabla p + \mu \nabla^2 \mathbf{v} + \rho \mathbf{g}]$$

Cartesian coordinates (x, y, z):

$$\rho \left(\frac{\partial v_x}{\partial t} + v_x \frac{\partial v_x}{\partial x} + v_y \frac{\partial v_x}{\partial y} + v_z \frac{\partial v_x}{\partial z} \right) = -\frac{\partial p}{\partial x} + \mu \left[\frac{\partial^2 v_x}{\partial x^2} + \frac{\partial^2 v_x}{\partial y^2} + \frac{\partial^2 v_x}{\partial z^2} \right] + \rho g_x \quad (\text{B.6-1})$$

$$\rho \left(\frac{\partial v_y}{\partial t} + v_x \frac{\partial v_y}{\partial x} + v_y \frac{\partial v_y}{\partial y} + v_z \frac{\partial v_y}{\partial z} \right) = -\frac{\partial p}{\partial y} + \mu \left[\frac{\partial^2 v_y}{\partial x^2} + \frac{\partial^2 v_y}{\partial y^2} + \frac{\partial^2 v_y}{\partial z^2} \right] + \rho g_y \quad (\text{B.6-2})$$

$$\rho \left(\frac{\partial v_z}{\partial t} + v_x \frac{\partial v_z}{\partial x} + v_y \frac{\partial v_z}{\partial y} + v_z \frac{\partial v_z}{\partial z} \right) = -\frac{\partial p}{\partial z} + \mu \left[\frac{\partial^2 v_z}{\partial x^2} + \frac{\partial^2 v_z}{\partial y^2} + \frac{\partial^2 v_z}{\partial z^2} \right] + \rho g_z \quad (\text{B.6-3})$$

Cylindrical coordinates (r, θ, z):

$$\rho \left(\frac{\partial v_r}{\partial t} + v_r \frac{\partial v_r}{\partial r} + \frac{v_\theta}{r} \frac{\partial v_r}{\partial \theta} + v_z \frac{\partial v_r}{\partial z} - \frac{v_\theta^2}{r} \right) = -\frac{\partial p}{\partial r} + \mu \left[\frac{\partial}{\partial r} \left(\frac{1}{r} \frac{\partial}{\partial r} (r v_r) \right) + \frac{1}{r^2} \frac{\partial^2 v_r}{\partial \theta^2} + \frac{\partial^2 v_r}{\partial z^2} - \frac{2}{r^2} \frac{\partial v_\theta}{\partial \theta} \right] + \rho g_r \quad (\text{B.6-4})$$

$$\rho \left(\frac{\partial v_\theta}{\partial t} + v_r \frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r} \frac{\partial v_\theta}{\partial \theta} + v_z \frac{\partial v_\theta}{\partial z} + \frac{v_r v_\theta}{r} \right) = -\frac{1}{r} \frac{\partial p}{\partial \theta} + \mu \left[\frac{\partial}{\partial r} \left(\frac{1}{r} \frac{\partial}{\partial r} (r v_\theta) \right) + \frac{1}{r^2} \frac{\partial^2 v_\theta}{\partial \theta^2} + \frac{\partial^2 v_\theta}{\partial z^2} + \frac{2}{r^2} \frac{\partial v_r}{\partial \theta} \right] + \rho g_\theta \quad (\text{B.6-5})$$

$$\rho \left(\frac{\partial v_z}{\partial t} + v_r \frac{\partial v_z}{\partial r} + \frac{v_\theta}{r} \frac{\partial v_z}{\partial \theta} + v_z \frac{\partial v_z}{\partial z} \right) = -\frac{\partial p}{\partial z} + \mu \left[\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial v_z}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 v_z}{\partial \theta^2} + \frac{\partial^2 v_z}{\partial z^2} \right] + \rho g_z \quad (\text{B.6-6})$$

Spherical coordinates (r, θ, ϕ):

$$\rho \left(\frac{\partial v_r}{\partial t} + v_r \frac{\partial v_r}{\partial r} + \frac{v_\theta}{r} \frac{\partial v_r}{\partial \theta} + \frac{v_\phi}{r \sin \theta} \frac{\partial v_r}{\partial \phi} - \frac{v_\theta^2 + v_\phi^2}{r} \right) = -\frac{\partial p}{\partial r} + \mu \left[\frac{1}{r^2} \frac{\partial^2}{\partial r^2} (r^2 v_r) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial v_r}{\partial \theta} \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 v_r}{\partial \phi^2} \right] + \rho g_r \quad (\text{B.6-7})^a$$

$$\rho \left(\frac{\partial v_\theta}{\partial t} + v_r \frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r} \frac{\partial v_\theta}{\partial \theta} + \frac{v_\phi}{r \sin \theta} \frac{\partial v_\theta}{\partial \phi} + \frac{v_r v_\theta - v_\phi^2 \cot \theta}{r} \right) = -\frac{1}{r} \frac{\partial p}{\partial \theta} + \mu \left[\frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial v_\theta}{\partial r} \right) + \frac{1}{r^2} \frac{\partial}{\partial \theta} \left(\frac{1}{\sin \theta} \frac{\partial}{\partial \theta} (v_\theta \sin \theta) \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 v_\theta}{\partial \phi^2} + \frac{2}{r^2} \frac{\partial v_r}{\partial \theta} - \frac{2 \cot \theta}{r^2 \sin \theta} \frac{\partial v_\phi}{\partial \phi} \right] + \rho g_\theta \quad (\text{B.6-8})$$

$$\rho \left(\frac{\partial v_\phi}{\partial t} + v_r \frac{\partial v_\phi}{\partial r} + \frac{v_\theta}{r} \frac{\partial v_\phi}{\partial \theta} + \frac{v_\phi}{r \sin \theta} \frac{\partial v_\phi}{\partial \phi} + \frac{v_\phi v_r + v_\theta v_\phi \cot \theta}{r} \right) = -\frac{1}{r \sin \theta} \frac{\partial p}{\partial \phi} + \mu \left[\frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial v_\phi}{\partial r} \right) + \frac{1}{r^2} \frac{\partial}{\partial \theta} \left(\frac{1}{\sin \theta} \frac{\partial}{\partial \theta} (v_\phi \sin \theta) \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 v_\phi}{\partial \phi^2} + \frac{2}{r^2 \sin \theta} \frac{\partial v_r}{\partial \phi} + \frac{2 \cot \theta}{r^2 \sin \theta} \frac{\partial v_\theta}{\partial \phi} \right] + \rho g_\phi \quad (\text{B.6-9})$$

^a The quantity in the brackets in Eq. B.6-7 is *not* what one would expect from Eq. (M) for $[\nabla \cdot \nabla \mathbf{v}]$ in Table A.7-3, because we have added to Eq. (M) the expression for $(2/r)(\nabla \cdot \mathbf{v})$, which is zero for fluids with constant ρ . This gives a much simpler equation.

§B.7 THE DISSIPATION FUNCTION Φ_v FOR NEWTONIAN FLUIDS (SEE EQ. 3.3-3)

Cartesian coordinates (x, y, z):

$$\Phi_v = 2 \left[\left(\frac{\partial v_x}{\partial x} \right)^2 + \left(\frac{\partial v_y}{\partial y} \right)^2 + \left(\frac{\partial v_z}{\partial z} \right)^2 \right] + \left[\frac{\partial v_y}{\partial x} + \frac{\partial v_x}{\partial y} \right]^2 + \left[\frac{\partial v_z}{\partial y} + \frac{\partial v_y}{\partial z} \right]^2 + \left[\frac{\partial v_x}{\partial z} + \frac{\partial v_z}{\partial x} \right]^2 - \frac{2}{3} \left[\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z} \right]^2 \quad (\text{B.7-1})$$

Cylindrical coordinates (r, θ , z):

$$\begin{aligned} \Phi_v = 2 & \left[\left(\frac{\partial v_r}{\partial r} \right)^2 + \left(\frac{1}{r} \frac{\partial v_\theta}{\partial \theta} + \frac{v_r}{r} \right)^2 + \left(\frac{\partial v_z}{\partial z} \right)^2 \right] + \left[r \frac{\partial}{\partial r} \left(\frac{v_\theta}{r} \right) + \frac{1}{r} \frac{\partial v_r}{\partial \theta} \right]^2 + \left[\frac{1}{r} \frac{\partial v_z}{\partial \theta} + \frac{\partial v_\theta}{\partial z} \right]^2 + \left[\frac{\partial v_r}{\partial z} + \frac{\partial v_z}{\partial r} \right]^2 \\ & - \frac{2}{3} \left[\frac{1}{r} \frac{\partial}{\partial r} (rv_r) + \frac{1}{r} \frac{\partial v_\theta}{\partial \theta} + \frac{\partial v_z}{\partial z} \right]^2 \end{aligned} \quad (\text{B.7-2})$$

Spherical coordinates (r, θ , ϕ):

$$\begin{aligned} \Phi_v = 2 & \left[\left(\frac{\partial v_r}{\partial r} \right)^2 + \left(\frac{1}{r} \frac{\partial v_\theta}{\partial \theta} + \frac{v_r}{r} \right)^2 + \left(\frac{1}{r \sin \theta} \frac{\partial v_\phi}{\partial \phi} + \frac{v_r + v_\theta \cot \theta}{r} \right)^2 \right] \\ & + \left[r \frac{\partial}{\partial r} \left(\frac{v_\theta}{r} \right) + \frac{1}{r} \frac{\partial v_r}{\partial \theta} \right]^2 + \left[\frac{\sin \theta}{r} \frac{\partial}{\partial \theta} \left(\frac{v_\phi}{\sin \theta} \right) + \frac{1}{r \sin \theta} \frac{\partial v_\theta}{\partial \phi} \right]^2 + \left[\frac{1}{r \sin \theta} \frac{\partial v_r}{\partial \phi} + r \frac{\partial}{\partial r} \left(\frac{v_\phi}{r} \right) \right]^2 \\ & - \frac{2}{3} \left[\frac{1}{r^2} \frac{\partial}{\partial r} (r^2 v_r) + \frac{1}{r \sin \theta} \frac{\partial}{\partial \theta} (v_\theta \sin \theta) + \frac{1}{r \sin \theta} \frac{\partial v_\phi}{\partial \phi} \right]^2 \end{aligned} \quad (\text{B.7-3})$$

§B.8 THE EQUATION OF ENERGY IN TERMS OF q

$$[\rho \hat{C}_p DT/Dt = -(\nabla \cdot \mathbf{q}) - (\partial \ln \rho / \partial \ln T)_p Dp/Dt - (\boldsymbol{\tau} : \nabla \mathbf{v})]$$

Cartesian coordinates (x, y, z):

$$\rho \hat{C}_p \left(\frac{\partial T}{\partial t} + v_x \frac{\partial T}{\partial x} + v_y \frac{\partial T}{\partial y} + v_z \frac{\partial T}{\partial z} \right) = - \left[\frac{\partial q_x}{\partial x} + \frac{\partial q_y}{\partial y} + \frac{\partial q_z}{\partial z} \right] - \left(\frac{\partial \ln \rho}{\partial \ln T} \right)_p \frac{Dp}{Dt} - (\boldsymbol{\tau} : \nabla \mathbf{v}) \quad (\text{B.8-1})^a$$

Cylindrical coordinates (r, θ , z):

$$\rho \hat{C}_p \left(\frac{\partial T}{\partial t} + v_r \frac{\partial T}{\partial r} + \frac{v_\theta}{r} \frac{\partial T}{\partial \theta} + v_z \frac{\partial T}{\partial z} \right) = - \left[\frac{1}{r} \frac{\partial}{\partial r} (rq_r) + \frac{1}{r} \frac{\partial q_\theta}{\partial \theta} + \frac{\partial q_z}{\partial z} \right] - \left(\frac{\partial \ln \rho}{\partial \ln T} \right)_p \frac{Dp}{Dt} - (\boldsymbol{\tau} : \nabla \mathbf{v}) \quad (\text{B.8-2})^a$$

Spherical coordinates (r, θ , ϕ):

$$\rho \hat{C}_p \left(\frac{\partial T}{\partial t} + v_r \frac{\partial T}{\partial r} + \frac{v_\theta}{r} \frac{\partial T}{\partial \theta} + \frac{v_\phi}{r \sin \theta} \frac{\partial T}{\partial \phi} \right) = \left[\frac{1}{r^2} \frac{\partial}{\partial r} (r^2 q_r) + \frac{1}{r \sin \theta} \frac{\partial}{\partial \theta} (q_\theta \sin \theta) + \frac{1}{r \sin \theta} \frac{\partial q_\phi}{\partial \phi} \right] - \left(\frac{\partial \ln \rho}{\partial \ln T} \right)_p \frac{Dp}{Dt} - (\boldsymbol{\tau} : \nabla \mathbf{v}) \quad (\text{B.8-3})^a$$

^a The viscous dissipation term, $-(\boldsymbol{\tau} : \nabla \mathbf{v})$, is given in Appendix A, Tables A.7-1, 2, 3. This term may usually be neglected, except for systems with very large velocity gradients. The term containing $(\partial \ln \rho / \partial \ln T)_p$ is zero for fluid with constant ρ .

§B.9 THE EQUATION OF ENERGY FOR PURE NEWTONIAN FLUIDS WITH CONSTANT^a ρ AND k

$$[\rho \hat{C}_p DT/Dt = k \nabla^2 T + \mu \Phi_v]$$

Cartesian coordinates (x, y, z):

$$\rho \hat{C}_p \left(\frac{\partial T}{\partial t} + v_x \frac{\partial T}{\partial x} + v_y \frac{\partial T}{\partial y} + v_z \frac{\partial T}{\partial z} \right) = k \left[\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} \right] + \mu \Phi_v \quad (\text{B.9-1})^b$$

Cylindrical coordinates (r, θ, z):

$$\rho \hat{C}_p \left(\frac{\partial T}{\partial t} + v_r \frac{\partial T}{\partial r} + \frac{v_\theta}{r} \frac{\partial T}{\partial \theta} + v_z \frac{\partial T}{\partial z} \right) = k \left[\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial T}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 T}{\partial \theta^2} + \frac{\partial^2 T}{\partial z^2} \right] + \mu \Phi_v \quad (\text{B.9-2})^b$$

Spherical coordinates (r, θ, ϕ):

$$\rho \hat{C}_p \left(\frac{\partial T}{\partial t} + v_r \frac{\partial T}{\partial r} + \frac{v_\theta}{r} \frac{\partial T}{\partial \theta} + \frac{v_\phi}{r \sin \theta} \frac{\partial T}{\partial \phi} \right) = k \left[\frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial T}{\partial r} \right) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial T}{\partial \theta} \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 T}{\partial \phi^2} \right] + \mu \Phi_v \quad (\text{B.9-3})^b$$

^a This form of the energy equation is also valid under the less stringent assumptions $k = \text{constant}$ and $(\partial \ln \rho / \partial \ln T)_p Dp/Dt = 0$. The assumption $\rho = \text{constant}$ is given in the table heading because it is the assumption more often made.

^b The function Φ_v is given in §B.7. The term $\mu \Phi_v$ is usually negligible, except in systems with large velocity gradients.

§B.10 THE EQUATION OF CONTINUITY FOR SPECIES α IN TERMS^a OF \mathbf{j}_α

$$[\rho D\omega_\alpha/Dt = -(\nabla \cdot \mathbf{j}_\alpha) + r_\alpha]$$

Cartesian coordinates (x, y, z):

$$\rho \left(\frac{\partial \omega_\alpha}{\partial t} + v_x \frac{\partial \omega_\alpha}{\partial x} + v_y \frac{\partial \omega_\alpha}{\partial y} + v_z \frac{\partial \omega_\alpha}{\partial z} \right) = - \left[\frac{\partial j_{\alpha x}}{\partial x} + \frac{\partial j_{\alpha y}}{\partial y} + \frac{\partial j_{\alpha z}}{\partial z} \right] + r_\alpha \quad (\text{B.10-1})$$

Cylindrical coordinates (r, θ, z):

$$\rho \left(\frac{\partial \omega_\alpha}{\partial t} + v_r \frac{\partial \omega_\alpha}{\partial r} + \frac{v_\theta}{r} \frac{\partial \omega_\alpha}{\partial \theta} + v_z \frac{\partial \omega_\alpha}{\partial z} \right) = - \left[\frac{1}{r} \frac{\partial}{\partial r} (r j_{\alpha r}) + \frac{1}{r} \frac{\partial j_{\alpha \theta}}{\partial \theta} + \frac{\partial j_{\alpha z}}{\partial z} \right] + r_\alpha \quad (\text{B.10-2})$$

Spherical coordinates (r, θ, ϕ):

$$\rho \left(\frac{\partial \omega_\alpha}{\partial t} + v_r \frac{\partial \omega_\alpha}{\partial r} + \frac{v_\theta}{r} \frac{\partial \omega_\alpha}{\partial \theta} + \frac{v_\phi}{r \sin \theta} \frac{\partial \omega_\alpha}{\partial \phi} \right) = \left[\frac{1}{r^2} \frac{\partial}{\partial r} (r^2 j_{\alpha r}) + \frac{1}{r \sin \theta} \frac{\partial}{\partial \theta} (j_{\alpha \theta} \sin \theta) + \frac{1}{r \sin \theta} \frac{\partial j_{\alpha \phi}}{\partial \phi} \right] + r_\alpha \quad (\text{B.10-3})$$

^a To obtain the corresponding equations in terms of \mathbf{J}_α^* make the following replacements:

Replace	ρ	ω_α	\mathbf{j}_α	\mathbf{v}	r_α
by	c	x_α	\mathbf{J}_α^*	\mathbf{v}^*	$R_\alpha - x_\alpha \sum_{\beta=1}^N R_\beta$

**§B.11 THE EQUATION OF CONTINUITY FOR SPECIES A
IN TERMS OF ω_A FOR CONSTANT^a $\rho_{\mathcal{D}_{AB}}$**

$$[\rho D\omega_A/Dt = \rho_{\mathcal{D}_{AB}} \nabla^2 \omega_A + r_A]$$

Cartesian coordinates (x, y, z):

$$\rho \left(\frac{\partial \omega_A}{\partial t} + v_x \frac{\partial \omega_A}{\partial x} + v_y \frac{\partial \omega_A}{\partial y} + v_z \frac{\partial \omega_A}{\partial z} \right) = \rho_{\mathcal{D}_{AB}} \left[\frac{\partial^2 \omega_A}{\partial x^2} + \frac{\partial^2 \omega_A}{\partial y^2} + \frac{\partial^2 \omega_A}{\partial z^2} \right] + r_A \quad (\text{B.11-1})$$

Cylindrical coordinates (r, θ , z):

$$\rho \left(\frac{\partial \omega_A}{\partial t} + v_r \frac{\partial \omega_A}{\partial r} + \frac{v_\theta}{r} \frac{\partial \omega_A}{\partial \theta} + v_z \frac{\partial \omega_A}{\partial z} \right) = \rho_{\mathcal{D}_{AB}} \left[\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial \omega_A}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 \omega_A}{\partial \theta^2} + \frac{\partial^2 \omega_A}{\partial z^2} \right] + r_A \quad (\text{B.11-2})$$

Spherical coordinates (r, θ , ϕ):

$$\rho \left(\frac{\partial \omega_A}{\partial t} + v_r \frac{\partial \omega_A}{\partial r} + \frac{v_\theta}{r} \frac{\partial \omega_A}{\partial \theta} + \frac{v_\phi}{r \sin \theta} \frac{\partial \omega_A}{\partial \phi} \right) = \rho_{\mathcal{D}_{AB}} \left[\frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial \omega_A}{\partial r} \right) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial \omega_A}{\partial \theta} \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 \omega_A}{\partial \phi^2} \right] + r_A \quad (\text{B.11-3})$$

^a To obtain the corresponding equations in terms of x_α , make the following replacements:

Replace	ρ	ω_α	\mathbf{v}	r_α
by	c	x_α	\mathbf{v}^*	$R_\alpha - x_\alpha \sum_{\beta=1}^N R_\beta$