3B.10: Radial flow of a Newtonian fluid between parallel disks (BLS, page 108)

A part of a lubrication system consists of two circular disks between which a lubricant flows radially. The flow takes place because of a modified pressure difference ($P_1 - P_2$) between the inner and outer radii r_1 and r_2,respectively.

Steady, laminar flow occurs in the space between two fixed parallel, circular disks separated by a small gap $2b$. The fluid flows radially outward owing to a pressure difference ($P_1 - P_2$) between the inner and outer radii r_1 and r_2, respectively. Neglect end effects and consider the region $r_1 \le r \le r_2$ only. Such a flow occurs when a lubricant flows in certain lubrication systems (It means high viscosity, and so low Reynolds number, Re~1).

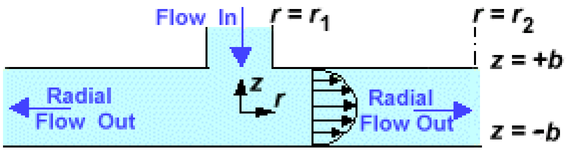

Figure. Radial flow between two parallel disks.

a) Simplify the equation of continuity to show that $r\,v_r = f$, where f is a function of only z.

b) Simplify the equation of motion for incompressible flow of a Newtonian fluid of viscosity μ and density ρ.

c) Obtain the velocity profile assuming creeping flow.

d) Sketch the velocity profile $v_r\,(r, z)$ and the pressure profile $P(r)$.

e) Determine an expression for the mass flow rate by integrating the velocity profile.

Solution

Step. Simplification of continuity equation:

Since the steady laminar flow is directed radially outward, only the radial velocity component v_r exists. The tangential and axial components of velocity are zero; so, $v_\theta = 0$ and $v_z = 0$.

For incompressible flow, the continuity equation gives $\nabla.\mathbf{v} = 0$.

In cylindrical coordinates,

$$\frac{1}{r}\frac{\partial}{\partial r}(r\,v_r) + \frac{1}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{\partial v_z}{\partial z} = 0 \quad \Rightarrow \quad \frac{\partial}{\partial r}(r\,v_r) = 0 \qquad (1)$$

On integrating the simplified continuity equation, $r\,v_r = f(\theta, z)$. Since the solution is expected to be symmetric about the z-axis, there is no dependence on the angle θ. Thus, f is a function of z only and not of r or θ. In other words, $r\, v_r = f(z)$. This is simply explained from the fact that mass (or volume, if density ρ is constant) is conserved; so, $\rho\,(2\,\pi\, r\, v_r\, dz) = dw$ is constant (at a given z) and is independent of r.

Step. For a Newtonian fluid, the Navier - Stokes equation is

$$\rho\,\frac{D\mathbf{v}}{Dt} = -\nabla P + \mu\nabla^2\mathbf{v} \qquad (2)$$

in which P includes both the pressure and gravitational terms. On noting that $v_r = v_r(r, z)$, its components for steady flow in cylindrical coordinates may be simplified as given below.

r - component

$$\rho\left(v_r\frac{\partial v_r}{\partial r}\right) = -\frac{\partial P}{\partial r} + \mu\left[\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(r\, v_r)\right) + \frac{\partial^2 v_r}{\partial z^2}\right] \qquad (3)$$

Θ - component :

$$0 = \frac{\partial P}{\partial \theta} \qquad (4)$$

z - component :

$$0 = \frac{\partial P}{\partial z} \qquad (5)$$

Recall that $r\, v_r = f(z)$ from the continuity equation. Substituting $v_r = f\,/\,r$ and $P = P(r)$ in equation (3) then gives

$$-\rho\,\frac{f^2}{r^3} = -\frac{dP}{dr} + \frac{\mu}{r}\frac{d^2 f}{dz^2} \qquad (6)$$

Equation (6) has no solution unless the nonlinear term (that is, the f^2 term on the left-hand side) is neglected. Under this 'creeping flow' assumption, equation (6) may be written as

$$r\,\frac{dP}{dr} = \mu\,\frac{d^2 f}{dz^2} \qquad (7)$$

The left-hand side of equation (7) is a function of r only, whereas the right-hand side is a function of z only. This is only possible if each side equals a constant (say, C_0). Integration with respect to r from the inner radius r_1 to the outer radius r_2 then gives $P_2 - P_1 = C_0 \ln(r_2/r_1)$. On replacing C_0 in terms of f,

$$0 = (P_1 - P_2) + \left(\mu \ln\frac{r_2}{r_1}\right)\frac{d^2 f}{dz^2} \qquad (8)$$

The above equation may be integrated twice with respect to z as follows.

$$\frac{df}{dz} = \frac{-\Delta P}{\mu \ln (r_2/r_1)} z + C_1 \quad (9)$$

$$f = \frac{-\Delta P}{2\,\mu \ln (r_2/r_1)} z^2 + C_1 z + C_2$$

$$\Rightarrow \quad v_r = \frac{-\Delta P}{2\,\mu\, r \ln (r_2/r_1)} z^2 + C_1 \frac{z}{r} + \frac{C_2}{r} \quad (10)$$

Here, $\Delta P \equiv P_1 - P_2$. Equation (10) is valid in the region $r_1 \leq r \leq r_2$ and $-b \leq z \leq b$.

Imposing the no-slip boundary conditions at the two stationary disk surfaces ($v_z = 0$ at $z = \pm b$ and any r) gives $C_1 = 0$ and $C_2 = \Delta P\, b^2 / [2\,\mu \ln (r_2/r_1)]$. On substituting the integration constants in equation (10), the velocity profile is ultimately obtained as

$$v_r = \frac{\Delta P\, b^2}{2\,\mu\, r \ln (r_2/r_1)} \left[1 - \left(\frac{z}{b} \right)^2 \right] \quad (11)$$

Step. Sketch of velocity profile and pressure profile

The velocity profile from equation (11) is observed to be parabolic for each value of r with $v_{r,max} = \Delta P\, b^2 / [2\,\mu\, r \ln (r_2/r_1)]$. The maximum velocity at $z = 0$ is thus inversely proportional to r. In general, it is observed from equation (11) that v_r itself is inversely proportional to r. Sketches of $v_r(z)$ for different values of r and $v_r(r)$ for different values of $|z|$ may be plotted.

The pressure profile obtained by integrating the left-hand side of equation (7) is $(P - P_2) / (P_1 - P_2) = [\ln(r/r_2)] / [\ln(r_1/r_2)]$. A sketch of $P(r)$ may be plotted which holds for all z.

Step. Mass flow rate by integrating velocity profile

The mass flow rate w is rigorously obtained by integrating the velocity profile using $w = \int \mathbf{n} \cdot \rho \mathbf{v}\, dS$, where $\mathbf{n}$ is the unit normal to the element of surface area dS and $\mathbf{v}$ is the fluid velocity vector. For the radial flow between parallel disks, $\mathbf{n} = \boldsymbol{\delta}_r$, $\mathbf{v} = v_r \boldsymbol{\delta}_r$, and $dS = 2\pi r\, dz$. Then, substituting the velocity profile from equation (11) and integrating gives

$$w = \int_{-b}^{b} \rho\, v_r\, (2\,\pi\, r)\, dz = \frac{\pi\, \Delta P\, b^2 \rho}{\mu \ln (r_2/r_1)} \left[z - \frac{z^3}{3b^2} \right]_{-b}^{b} = \frac{4\pi\, \Delta P\, b^3 \rho}{3\mu \ln (r_2/r_1)} \quad (12)$$