Newtonian fluid flow in a slightly tapered tube

A fluid (of constant density \( \rho \)) is in incompressible, laminar flow through a tube of length \( L \). The radius of the tube of circular cross section changes linearly from \( R_0 \) at the tube entrance \((z = 0)\) to a slightly smaller value \( R_L \) at the tube exit \((z = L)\).

Such a flow occurs when a lubricant flows in certain lubrication systems (It means high viscosity, and so low Reynolds number, \( \text{Re} \sim 1 \)).

![Figure. Fluid flow in a slightly tapered tube.](image)

Using the lubrication approximation, determine the mass flow rate vs. pressure drop \((w \text{ vs. } \Delta P)\) relationship for a Newtonian fluid (of constant viscosity \( \mu \)).

Solution

Step. Hagen-Poiseuille equation and lubrication approximation:

The mass flow rate vs. pressure drop \((w \text{ vs. } \Delta P)\) relationship for a Newtonian fluid in a circular tube of constant radius \( R \) is

\[
   w = \frac{\pi \Delta P R^4 \rho}{8 \mu L} \quad (1)
\]

The above equation, which is the famous Hagen-Poiseuille equation, may be re-arranged as

\[
   \frac{\Delta P}{L} = \frac{8 \mu w \rho}{\pi R^4} \quad (2)
\]

For the tapered tube, note that the mass flow rate \( w \) does not change with axial distance \( z \). If the above equation is assumed to be approximately valid for a differential length \( dz \) of the tube whose radius \( R \) is slowly changing with axial distance \( z \), then it may be re-written as

\[
   -\frac{dP}{dz} = \frac{8 \mu w}{\rho \pi} \frac{1}{[R(z)]^4} \quad (3)
\]

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The approximation used above where a flow between non-parallel surfaces is treated locally as a flow between parallel surfaces is commonly called the lubrication approximation because it is often employed in the theory of lubrication. The lubrication approximation, simply speaking, is a local application of a one-dimensional solution and therefore may be referred to as a quasi-one-dimensional approach.

Equation (3) may be integrated to obtain the pressure drop across the tube on substituting the taper function $R(z)$, which is determined next.

**Step. Taper function**

As the tube radius $R$ varies linearly from $R_0$ at the tube entrance ($z = 0$) to $R_L$ at the tube exit ($z = L$), the taper function may be expressed as $R(z) = R_0 + (R_L - R_0) z / L$. On differentiating with respect to $z$, we get

$$
\frac{dR}{dz} = \frac{R_L - R_0}{L} \quad (4)
$$

Equation (3) is readily integrated with respect to radius $R$ rather than axial distance $z$. Using equation (4) to eliminate $dz$ from equation (3) yields

$$
(-dP) = \frac{8 \mu w}{\rho \pi} \frac{L}{R_L - R_0} \frac{dR}{R^4} \quad (5)
$$

Integrating the above equation between $z = 0$ and $z = L$, we get

$$
\frac{P_L}{P_0} = \int (-dP) = \frac{8 \mu w}{\rho \pi} \frac{L}{R_L - R_0} \int \frac{R_L dR}{R^4} \quad (6)
$$

$$
\frac{P_0 - P_L}{L} = \frac{8 \mu w}{\rho \pi} \frac{1}{3 (R_L - R_0)} \left( \frac{1}{R_0^3} - \frac{1}{R_L^3} \right) \quad (7)
$$

Equation (7) may be re-arranged into the following standard form in terms of mass flow rate:

$$
w = \frac{\pi \Delta P}{8 \mu L} R_0^4 \rho \left[ \frac{3 (\lambda - 1)}{1 - \lambda^3} \right] = \frac{\pi \Delta P}{8 \mu L} R_0^4 \rho \left[ \frac{3 \lambda^3}{1 + \lambda + \lambda^2} \right] \quad (8)
$$

where the taper ratio $\lambda \equiv R_L / R_0$. The term in square brackets on the right-hand side of the above equation may be viewed as a taper correction to equation (1).