$$\frac{D}{Dt}\rho = \underset{\sim}{\nabla} \cdot \rho \underset{\sim}{v}$$

$$\frac{D}{Dt}\rho \underset{\sim}{v} = -\underset{\sim}{\nabla} P - \underset{\sim}{\nabla} \cdot \underset{\approx}{\tau} + \rho \underset{\sim}{g}$$

Eq. di continuità, Eq. del moto

Moti bidimensionali, Funzione di Flusso

$$\left.\begin{aligned} v_x &= v_x(x,y,t) \\ v_y &= v_y(x,y,t) \\ P &= P(x,y,t) \end{aligned}\right\} incognite$$

$$\left.\begin{aligned} \rho \frac{D}{Dt} v_x &= -\frac{\partial}{\partial x}\wp + \mu\left(\frac{\partial^2}{\partial x^2} v_x + \frac{\partial^2}{\partial y^2} v_x\right) \\ \rho \frac{D}{Dt} v_y &= -\frac{\partial}{\partial y}\wp + \mu\left(\frac{\partial^2}{\partial x^2} v_y + \frac{\partial^2}{\partial y^2} v_y\right) \\ \frac{\partial}{\partial x} v_x &+ \frac{\partial}{\partial y} v_y = 0 \end{aligned}\right\} equazioni$$

Eq. del moto per μ=cost, Eq. di continuità per ρ=cost

$$\frac{\partial}{\partial y}\left(\rho \frac{D}{Dt} v_x\right) = \frac{\partial}{\partial y}\left(-\frac{\partial}{\partial x}\wp\right) + \mu \frac{\partial}{\partial y}\left(\frac{\partial^2}{\partial x^2} v_x + \frac{\partial^2}{\partial y^2} v_x\right)$$

$$\frac{\partial}{\partial x}\left(\rho \frac{D}{Dt} v_y\right) = \frac{\partial}{\partial x}\left(-\frac{\partial}{\partial y}\wp\right) + \mu \frac{\partial}{\partial x}\left(\frac{\partial^2}{\partial x^2} v_y + \frac{\partial^2}{\partial y^2} v_y\right)$$

sottraendo si elimina la pressione

$$\rho \frac{D}{Dt}\left(\frac{\partial}{\partial y} v_x - \frac{\partial}{\partial x} v_y\right) = \mu\left(\frac{\partial^2}{\partial x^2}\frac{\partial}{\partial y} v_x + \frac{\partial^2}{\partial y^2}\frac{\partial}{\partial y} v_x - \frac{\partial^2}{\partial x^2}\frac{\partial}{\partial x} v_y - \frac{\partial^2}{\partial y^2}\frac{\partial}{\partial x} v_y\right)$$

$$v_x = -\frac{\partial}{\partial y}\psi, v_y = +\frac{\partial}{\partial x}\psi \quad \frac{\partial}{\partial x}\left(-\frac{\partial}{\partial y}\psi\right) + \frac{\partial}{\partial y}\left(+\frac{\partial}{\partial x}\psi\right) = 0$$

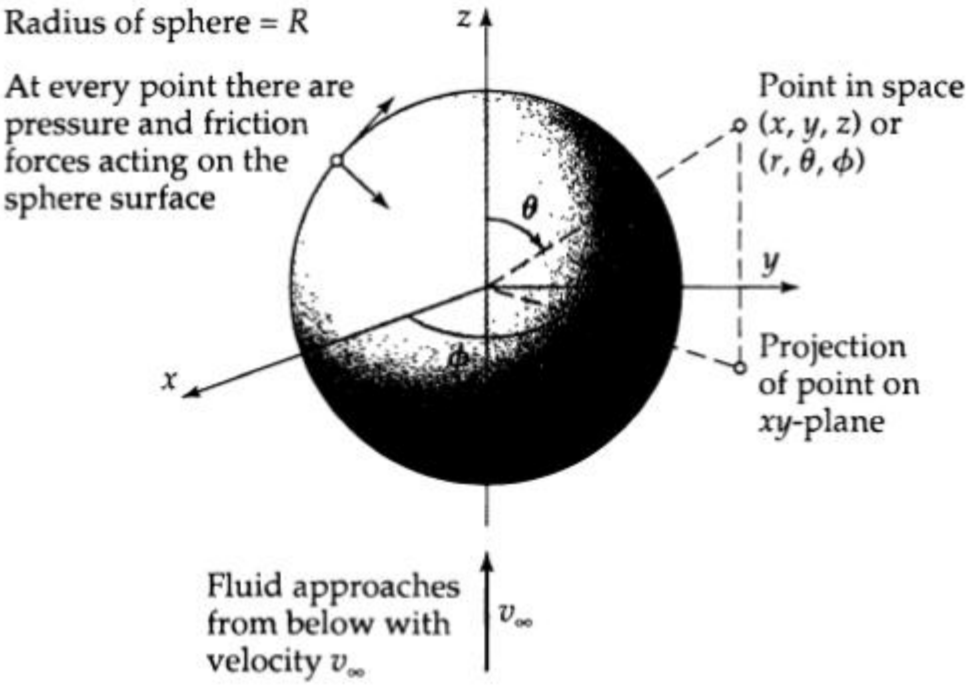
soddisfa Eq. di continuità

$$\rho \frac{D}{Dt}\left(\frac{\partial^2}{\partial y^2}\psi + \frac{\partial^2}{\partial x^2}\psi\right) = \mu\left(2\frac{\partial^2}{\partial x^2}\frac{\partial^2}{\partial y^2}\psi + \frac{\partial^2}{\partial y^2}\frac{\partial^2}{\partial y^2}\psi + \frac{\partial^2}{\partial x^2}\frac{\partial^2}{\partial x^2}\psi\right)$$

$$\rho \frac{D}{Dt}\left(\nabla^2\psi\right) = \mu\nabla^2\left(\nabla^2\psi\right) = \mu\nabla^4\psi$$

Moto viscoso intorno a una sfera (Re<1), bidimensionale in (r,θ), $v_\phi=0$

Creeping flow around a sphere 2.6 BLS pag.58



$$E^4\psi = E^2\left(E^2\psi\right) = 0$$

$$E^2 \equiv \frac{\partial^2}{\partial r^2} + \frac{\sin\theta}{r^2}\frac{\partial}{\partial\theta}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\right)$$

$$\begin{cases} r \to \infty \ \underset{\sim}{v} = \underset{\sim}{v}_\infty \\ r = R \ \underset{\sim}{v} = \underset{\sim}{0} \end{cases}$$

$$v_r = -\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\psi, v_\theta = \frac{1}{r\sin\theta}\frac{\partial}{\partial r}\psi$$

$$\begin{cases} r \to \infty \\ v_\theta = \dfrac{1}{r\sin\theta}\dfrac{\partial}{\partial r}\psi = -v_\infty \sin\theta \\ \psi(r,\theta) = -v_\infty \dfrac{r^2}{2}\sin^2\theta + c_1(\theta) \end{cases}$$

$$\begin{cases} r \to \infty \\ v_r = -\dfrac{1}{r^2 \sin\theta} \dfrac{\partial}{\partial\theta} \psi = v_\infty \cos\theta \\ \psi(r,\theta) = -v_\infty r^2 \dfrac{\sin^2\theta}{2} + c_2(r) \end{cases}$$

$$\begin{cases} r \to \infty \\ \psi(r,\theta) = -v_\infty \dfrac{r^2}{2} \sin^2\theta \end{cases}$$

$Hp.$ $\psi(r,\theta) = f(r)\sin^2\theta$

$$E^4\psi = 0$$

$$E^2\psi = \frac{\partial^2}{\partial r^2}\left(f\sin^2\theta\right) + \frac{\sin\theta}{r^2}\frac{\partial}{\partial\theta}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(f\sin^2\theta\right)\right) = \sin^2\theta\frac{d^2}{dr^2}f - \frac{2\sin^2\theta}{r^2}f$$

$$E^2\psi = \sin^2\theta\left(\frac{d^2}{dr^2} - \frac{2}{r^2}\right)f$$

$$E^4\psi = \sin^2\theta\left(\frac{d^2}{dr^2} - \frac{2}{r^2}\right)\left(\frac{d^2}{dr^2} - \frac{2}{r^2}\right)f$$

$$E^4\psi = g(r)\sin^2\theta = 0 \quad \forall\theta, r \quad \Rightarrow g(r) = 0 \quad \forall r$$

$$\left(\frac{d^2}{dr^2} - \frac{2}{r^2}\right)\left(\frac{d^2}{dr^2} - \frac{2}{r^2}\right)f = 0 \quad \forall r$$

$Hp.$ $f(r) = r^n$

$$\frac{d}{dr}f = nr^{n-1}$$

$$\frac{d^2}{dr^2}f = n(n-1)r^{n-2}$$

$$\left(\frac{d^2}{dr^2}-\frac{2}{r^2}\right)\left(\frac{d^2}{dr^2}-\frac{2}{r^2}\right)f=\left(\frac{d^2}{dr^2}-\frac{2}{r^2}\right)\left(n(n-1)r^{n-2}-\frac{2r^n}{r^2}\right)=\left(\frac{d^2}{dr^2}-\frac{2}{r^2}\right)(n(n-1)-2)r^{n-2}$$

$$\frac{d}{dr}r^{n-2}=(n-2)r^{n-3}$$

$$\frac{d^2}{dr^2}r^{n-2}=(n-2)(n-3)r^{n-4}$$

$$\left(\frac{d^2}{dr^2}-\frac{2}{r^2}\right)\left(\frac{d^2}{dr^2}-\frac{2}{r^2}\right)f=(n(n-1)-2)\left((n-2)(n-3)r^{n-4}-\frac{2}{r^2}r^{n-2}\right)=$$

$$=(n(n-1)-2)((n-2)(n-3)-2)r^{n-4}\quad\forall\mathrm{r}$$

$$(n(n-1)-2)((n-2)(n-3)-2)=0$$

$$n=-1;n=2;n=1;n=4\quad r^{-1};r^2;r^1;r^4$$

$$\psi=\left(Ar^{-1}+Br^1+Cr^2+Dr^4\right)\sin^2\theta$$

$$\begin{cases}r\to\infty\\ \psi(r,\theta)=-v_\infty\dfrac{r^2}{2}\sin^2\theta\end{cases}$$

$$D=0$$

$$C=-v_\infty\frac{1}{2}$$

$$\psi=\left(Ar^{-1}+Br^1-v_\infty\frac{r^2}{2}\right)\sin^2\theta$$

$$r=R\ v_r=0\quad v_\theta=0$$

$$v_r=-\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\psi=-\frac{1}{r^2\sin\theta}\left(Ar^{-1}+Br^1-v_\infty\frac{r^2}{2}\right)2\sin\theta\cos\theta=$$

$$=-\left(Ar^{-3}+Br^{-1}-v_\infty\frac{1}{2}\right)2\cos\theta$$

$$v_r\big|_{r=R}=-\left(AR^{-3}+BR^{-1}-v_\infty\frac{1}{2}\right)2\cos\theta=0\quad\forall\theta$$

$$\left(AR^{-3} + BR^{-1} - v_\infty \frac{1}{2} \right) = 0$$

$$v_\theta = \frac{1}{r\sin\theta}\frac{\partial}{\partial r}\psi = \frac{1}{r\sin\theta}\sin^2\theta\left(-Ar^{-2} + B - v_\infty 2\frac{r}{2} \right) =$$

$$= \sin\theta\left(-Ar^{-3} + Br^{-1} - v_\infty \right)$$

$$v_\theta\big|_{r=R} = \sin\theta\left(-AR^{-3} + BR^{-1} - v_\infty \right) = 0 \quad \forall\theta$$

$$\left(-AR^{-3} + BR^{-1} - v_\infty \right) = 0$$

$$\begin{cases} \left(AR^{-3} + BR^{-1} - v_\infty \frac{1}{2} \right) + \left(-AR^{-3} + BR^{-1} - v_\infty \right) = 0 \\ \left(AR^{-3} + BR^{-1} - v_\infty \frac{1}{2} \right) - \left(-AR^{-3} + BR^{-1} - v_\infty \right) = 0 \end{cases}$$

$$\begin{cases} 2BR^{-1} - \frac{3}{2}v_\infty = 0 \\ 2AR^{-3} + \frac{1}{2}v_\infty = 0 \end{cases}$$

$$\begin{cases} B = \frac{3}{4}Rv_\infty \\ A = -\frac{1}{4}R^3 v_\infty \end{cases}$$

$$\psi = \left(-\frac{1}{4}R^3 v_\infty r^{-1} + \frac{3}{4}Rv_\infty r^1 - v_\infty \frac{r^2}{2} \right)\sin^2\theta$$

$$\psi = v_\infty R^2\left(-\frac{1}{4}\left(\frac{r}{R}\right)^{-1} + \frac{3}{4}\left(\frac{r}{R}\right) - \frac{1}{2}\left(\frac{r}{R}\right)^2 \right)\sin^2\theta$$

$$v_r = -\left(Ar^{-3} + Br^{-1} - v_\infty \frac{1}{2} \right) 2\cos\theta$$

$$v_r = -\left(-\frac{1}{4}R^3 v_\infty r^{-3} + \frac{3}{4}Rv_\infty r^{-1} - v_\infty \frac{1}{2} \right) 2\cos\theta$$

$$v_r = -v_\infty\left(-\frac{1}{2}\left(\frac{r}{R}\right)^{-3} + \frac{3}{2}\left(\frac{r}{R}\right)^{-1} - 1\right)\cos\theta = v_\infty \cos\theta\left(1 - \frac{3}{2}\left(\frac{r}{R}\right)^{-1} + \frac{1}{2}\left(\frac{r}{R}\right)^{-3}\right)$$

$$v_\theta = \sin\theta\left(- -\frac{1}{4}R^3 v_\infty r^{-3} + \frac{3}{4}R v_\infty r^{-1} - v_\infty\right)$$

$$v_\theta = v_\infty \sin\theta\left(+\frac{1}{4}\left(\frac{r}{R}\right)^{-3} + \frac{3}{4}\left(\frac{r}{R}\right)^{-1} - 1\right) = -v_\infty \sin\theta\left(1 - \frac{3}{4}\left(\frac{r}{R}\right)^{-1} - \frac{1}{4}\left(\frac{r}{R}\right)^{-3}\right)$$

Pressione

$$\left.\begin{aligned}
&0 = -\frac{\partial}{\partial r}\wp + \mu\left(\frac{\partial^2}{\partial r^2}v_r + \frac{2}{r}\frac{\partial}{\partial r}v_r - \frac{2v_r}{r^2} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}v_r + \frac{\cot\theta}{r^2}\frac{\partial}{\partial\theta}v_r - \frac{2}{r^2}\frac{\partial}{\partial\theta}v_\theta - \frac{2v_\theta\cot\theta}{r^2}\right)\\
&0 = -\frac{1}{r}\frac{\partial}{\partial\theta}\wp + \mu\left(\frac{\partial^2}{\partial r^2}v_\theta + \frac{2}{r}\frac{\partial}{\partial r}v_\theta - \frac{v_\theta}{r^2\sin^2\theta} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}v_\theta + \frac{\cot\theta}{r^2}\frac{\partial}{\partial\theta}v_\theta + \frac{2}{r^2}\frac{\partial}{\partial\theta}v_r\right)\\
&\frac{\partial}{\partial r}v_r + \frac{2v_r}{r} + \frac{1}{r}\frac{\partial}{\partial\theta}v_\theta + \frac{v_\theta\cot\theta}{r} = 0
\end{aligned}\right\}$$

$$v_r = v_\infty\left(1 - \frac{3}{2}\left(\frac{r}{R}\right)^{-1} + \frac{1}{2}\left(\frac{r}{R}\right)^{-3}\right)\cos\theta$$

$$\frac{\partial}{\partial\theta}v_r = -v_\infty\left(1 - \frac{3}{2}\left(\frac{r}{R}\right)^{-1} + \frac{1}{2}\left(\frac{r}{R}\right)^{-3}\right)\sin\theta = -v_r\tan\theta$$

$$\frac{\partial^2}{\partial\theta^2}v_r = -v_\infty\left(1 - \frac{3}{2}\left(\frac{r}{R}\right)^{-1} + \frac{1}{2}\left(\frac{r}{R}\right)^{-3}\right)\cos\theta = -v_r$$

$$0 = -\frac{\partial}{\partial r}\wp + \mu\left(\frac{\partial^2}{\partial r^2}v_r + \frac{2}{r}\frac{\partial}{\partial r}v_r - \frac{2v_r}{r^2} - \frac{1}{r^2}v_r - \frac{1}{r^2}v_r - \frac{2}{r^2}\frac{\partial}{\partial\theta}v_\theta - \frac{2v_\theta\cot\theta}{r^2}\right)$$

$$v_\theta = -v_\infty\sin\theta\left(1 - \frac{3}{4}\left(\frac{r}{R}\right)^{-1} - \frac{1}{4}\left(\frac{r}{R}\right)^{-3}\right)$$

$$\frac{\partial}{\partial\theta}v_\theta = -v_\infty\left(1 - \frac{3}{4}\left(\frac{r}{R}\right)^{-1} - \frac{1}{4}\left(\frac{r}{R}\right)^{-3}\right)\cos\theta = v_\theta\cot\theta$$

$$0=-\frac{\partial}{\partial r}\wp+\mu\left(\frac{\partial^2}{\partial r^2}v_r+\frac{2}{r}\frac{\partial}{\partial r}v_r-\frac{4v_r}{r^2}-\frac{2}{r^2}v_\theta\cot\theta-\frac{2v_\theta\cot\theta}{r^2}\right)$$

$$0=-\frac{\partial}{\partial r}\wp+\mu\left(\frac{\partial^2}{\partial r^2}v_r+\frac{2}{r}\frac{\partial}{\partial r}v_r-\frac{4}{r^2}\left(v_r+v_\theta\cot\theta\right)\right)$$

$$\left(v_r+v_\theta\cot\theta\right)=v_\infty\cos\theta\left(1-\frac{3}{2}\left(\frac{r}{R}\right)^{-1}+\frac{1}{2}\left(\frac{r}{R}\right)^{-3}\right)-v_\infty\cos\theta\left(1-\frac{3}{4}\left(\frac{r}{R}\right)^{-1}-\frac{1}{4}\left(\frac{r}{R}\right)^{-3}\right)$$

$$\left(v_r+v_\theta\cot\theta\right)=v_\infty\cos\theta\left(-\frac{3}{4}\left(\frac{r}{R}\right)^{-1}+\frac{3}{4}\left(\frac{r}{R}\right)^{-3}\right)$$

$$-\frac{4}{r^2}\left(v_r+v_\theta\cot\theta\right)=\frac{1}{R^2}v_\infty\cos\theta\left(3\left(\frac{r}{R}\right)^{-3}-3\left(\frac{r}{R}\right)^{-5}\right)$$

$$\frac{\partial}{\partial r}v_r=\frac{1}{R}\frac{\partial}{\partial\left(r/R\right)}v_r=\frac{1}{R}v_\infty\cos\theta\left(\frac{3}{2}\left(\frac{r}{R}\right)^{-2}-\frac{3}{2}\left(\frac{r}{R}\right)^{-4}\right)$$

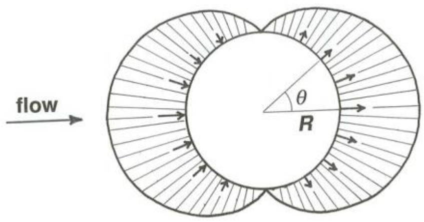
$$\frac{2}{r}\frac{\partial}{\partial r}v_r=\frac{1}{R}\frac{\partial}{\partial\left(r/R\right)}v_r=\frac{1}{R^2}v_\infty\cos\theta\left(3\left(\frac{r}{R}\right)^{-3}-3\left(\frac{r}{R}\right)^{-5}\right)$$

$$\frac{\partial^2}{\partial r^2}v_r=\frac{1}{R}\frac{\partial}{\partial\left(r/R\right)}\frac{1}{R}\frac{\partial}{\partial\left(r/R\right)}v_r=\frac{1}{R^2}v_\infty\cos\theta\left(-3\left(\frac{r}{R}\right)^{-3}+6\left(\frac{r}{R}\right)^{-5}\right)$$

$$\begin{cases}\dfrac{\partial}{\partial r}\wp=\mu\dfrac{1}{R^2}v_\infty\cos\theta\left(3\left(\dfrac{r}{R}\right)^{-3}\right)\\ r\to\infty\quad \wp=\wp_\infty\end{cases}$$

$$\wp\left(r,\theta\right)-\wp_\infty=\mu\frac{1}{R}v_\infty\cos\theta\left(-\frac{3}{2}\left(\frac{r}{R}\right)^{-2}\right)$$

La distribuzione delle pressioni sulla superficie della sfera è



$$\wp(R,\theta)-\wp_\infty=-\frac{3}{2}\mu\frac{1}{R}v_\infty\cos\theta$$

Forza di sollevamento

$$F_N=\int_0^{2\pi}\int_0^{\pi}\left(-\left(\wp+\tau_{rr}\right)\Big|_{r=R}\cos\theta\right)R^2\sin\theta d\theta d\phi=3\pi\mu R v_\infty\int_0^{\pi}\cos^2\theta\sin\theta d\theta=$$

$$=3\pi\mu R v_\infty\left(-\frac{1}{3}\cos^3\theta\Big|_0^{\pi}\right)=2\pi\mu R v_\infty$$

$$\tau_{rr}=-\mu\left(2\frac{\partial}{\partial r}v_r-\frac{2}{3}\underset{\sim}{\nabla}\cdot\underset{\sim}{v}\right)$$

$$\left(\tau_{rr}\right)\Big|_{r=R}=0$$

$$F_T=\int_0^{2\pi}\int_0^{\pi}\left(\tau_{r\theta}\Big|_{r=R}\sin\theta\right)R^2\sin\theta d\theta d\phi=2\pi\frac{3}{2}\mu R v_\infty\int_0^{\pi}\sin^3\theta d\theta=$$

$$=3\pi\mu R v_\infty\left(\frac{1}{12}\left(\cos(3\theta)-9\cos\theta\right)\Big|_0^{\pi}\right)=4\pi\mu R v_\infty$$

$$\tau_{r\theta}=-\mu\left(r\frac{\partial}{\partial r}\left(\frac{1}{r}v_\theta\right)+\frac{1}{r}\frac{\partial}{\partial\theta}\left(v_r\right)\right)$$

$$\left(\tau_{r\theta}\right)\Big|_{r=R}=\frac{3}{2}\mu\frac{1}{R}v_\infty\sin\theta$$

$$F=\frac{4}{3}\pi\rho g R^3+\underbrace{2\pi\mu R v_\infty+4\pi\mu R v_\infty}_{Stokes}$$