TRANSPORT PHENOMENA

A Unified Approach

Robert S. Brodkey
Harry C. Hershey

McGRAW-HILL INTERNATIONAL EDITIONS
Chemical Engineering Series

pressure drop across the boundary layer is negligible, this adverse gradient penetrates to the wall. The gradient in pressure, as well as the shear force, causes a rapid deceleration of the fluid elements in the immediate vicinity of the wall. The deceleration continues until the fluid elements come to rest, at which point the viscous forces are zero, since the velocity is zero. However, the adverse pressure effect will continue to act, and the fluid will reverse and flow backward. The point at which the reversal occurs is that at which $(dU_x/dy)_w$ is zero, and is the point of separation of the boundary layer from the surface. The separation must occur so that the flow can continue in the direction of the increasing pressure.

12.1.2 Tbe Turbulent Boundary Layer

The boundary layer for laminar flow was presented in Section 12.1.1 as a solution to the Navier-Stokes equations for flow over a flat plate. However, laminar flow will only exist over a short distance before transition occurs and turbulence 'is introduced. The flow in the laminar and turbulent boundary layers was illustrated in Fig. 6.2.' Along the front part of the plate a laminar boundary layer forms. When the Reynolds number, Eq. (6.3), is in the range of 5×10^5 to 5×10^6, the transition to turbulent flow begins. The transition can begin at lower Reynolds numbers in the presence of roughness or trip wires.' Once the transition occurs, there is a rapid change to the turbulent boundary layer as illustrated on the right of the figure. For each case, the velocity distribution is shown.

Equations (12.13) through (12.16) presented the relations for the stress at the wall, the total drag, and the drag coefficients for laminar boundary layers. Equation (12.19, relating the force or drag on a plate to the drag coefficient C_D, still applies:

$$F = \tfrac{1}{2}(\rho U_\infty^2)(Lb)(C_D) \qquad (12.20)$$

where the area A has been replaced by Lb. The drag coefficient C_D has been correlated empirically [S1]:

$$C_{D,\ \text{turbulent}} = 0.455[\log_{10}(N_{\text{Re},L})]^{-2.58} \qquad (12.21)$$

where the subscript "turbulent" indicates that Eq. (12.21) is restricted to the portion of the boundary layer that is turbulent. Similarly, the local shear stress at the wall is

$$\tau_w = \tfrac{1}{2}(\rho U_\infty^2)[2 \log_{10}(N_{\text{Re},x}) - 0.65]^{-2.3} \qquad (12.22)$$

If the turbulent velocity profile is approximated by the 1/7th-power law, Eq. (6.110), then the boundary layer thickness is

$$\delta/x = 0.376(N_{\text{Re},x})^{-0.2} \qquad (12.23)$$

[5] Film loop FM-5 illustrates the laminar and turbulent boundary layers and the tripping of a laminar layer.

TABLE 12.1
Variation of B with transition Reynolds number in Equation (12.24)

$N_{Re,c}$	B
3×10^5	1050
5×10^5	1700
1×10^6	3300
3×10^6	8700

* *Values taken* from Knudsen and Katz, *Fluid Dynamics and Heat Transfer.* p. 277, McGraw-Hill, New York, 1958.

Since Eq. (12.23) is based on the Blasius equation, Eq. (6.133), both are subject to the same restrictions.

The correlation in Eq. (12.21) applies to fully turbulent flow. Equation (12.21) can be empirically modified to account for the transition region and the effect of the laminar initial length [S1]:

$$C_D = 0.455[\log_{10}(N_{Re,L})]^{-2.58} - \frac{B}{N_{Re,L}} \tag{12.24}$$

where B is a function of the transition Reynolds number $N_{Re,c}$, as shown in Table 12.1. Here no subscript to denote conditions has been added since Eq. (12.24) applies over the entire region. If the point of transition is not known precisely, Eq. (12.24) will be approximate. The assumption of B equal to 1050 will be the most conservative, and a value of B equal to 1700 will be the most probable.

The total force obtained from Eqs. (12.20) and (12.21) assumes that the plate is subject to turbulent flow only. However, this condition is rarely encountered, and there is normally a laminar section preceding the turbulent region. Instead of using Eq. (12.24), an alternate procedure combines Eq. (12.14) for the force exerted in the laminar section and Eq. (12.20) for the force exerted in the turbulent section:

$$F_{actual} = F_{laminar,0\text{-}x_c} + F_{turbulent,0\text{-}L} - F_{turbulent,0\text{-}x_c} \tag{12.25}$$

$F_{laminar,0\text{-}x_c}$ evaluated from Eq. (12.14) from $x = 0$ to $x = x_c$ (at $N_{Re,c}$)

$F_{turbulent,0\text{-}L}$ evaluated from Eqs. (12.20) and (12.21) from $x = 0$ to $x = L$

$F_{turbulent,0\text{-}x_c}$ evaluated from Eqs. (12.20) and (12.21) from $x = 0$ to $x = x_c$

Here, the subscript $0-L$ refers to the entire length of plate and O-x, refers to the leading edge of the plate to x_c, the location of $N_{Re,c}$. Use of Eq. (12.25) will be illustrated in Example 12.3. Note that Eq. (12.24) included the laminar contribution when the parameter B was determined; thus Eqs. (12.24) and (12.25) cannot be used together.

Entry region for a pipe. For turbulent flow, the length in which the center line velocity reaches 99 percent of the maximum value is correlated by

$$L_e/d_o = 0.693(N_{Re})^{1/4} \tag{12.26}$$

Equation (12.26) is satisfactory for most normal problems where vibrations and flow disturbances result in a transition Reynolds number of approximately 2100. As a rule, Eq. (12.26) will compute a much shorter length L_e than will Eq. (12.19), the corresponding equation for the laminar entry in a pipe:

$$L_e/d_o = 0.0567 N_{Re} \tag{12.19}$$

Summary. The boundary layer equations of this and the previous section are summarized in Table 12.2.

TABLE 12.2
Summary of boundary layer equations

Description	Regime	Equation	
Boundary layer thickness	laminar	$\delta/x = 5.0(N_{Re,x})^{-1/2}$	(12.18)
	turbulent	$\delta/x = 0.376(N_{Re,x})^{-0.2}$	(12.23)
Wall shear stress	laminar	$\tau_w = -(\mu U_\infty C_1)[U_\infty/(\nu x)]^{1/2}$	(12.13)
	turbulent	$\tau_w = \frac{1}{2}(\rho U_\infty^2)[2\log_{10}(N_{Re,x}) - 0.65]^{-2.3}$	(12.22)
Total drag force	laminar	$F = (2C_1 U_\infty b)(\mu\rho L U_\infty)^{1/2}$	(12.14)
	turbulent	$F = \frac{1}{2}(\rho U_\infty^2)(Lb)(C_D)$	(12.20)
Local drag coefficient	laminar	$C_{D,x} = 2C_1(N_{Re,x})^{-1/2}$	(12.17)
Drag coefficient	laminar	$C_{D,\text{laminar}} = 4C_1(N_{Re,L})^{-1/2}$	(12.16)
	turbulent	$C_{D,\text{turbulent}} = 0.455[\log_{10}(N_{Re,L})]^{-2.58}$	(12.21)
	combined	$C_D = 0.455[\log_{10}(N_{Re,L})]^{-2.58} - B/(N_{Re,L})$	(12.24)
Pipe entry length	laminar	$L_e/d_o = 0.0567 N_{Re}$	(12.19)
	turbulent	$L_e/d_o = 0.693(N_{Re})^{1/4}$	(12.26)

Notes:

B	constant, cf. Table 12.1	L_e	entry length in a pipe
b	plate width	$N_{Re,x}$	equal to $xU_\infty\rho/\mu$
C_1	value of 0.33206	x	distance from leading edge
d_o	diameter of pipe	U_∞	free stream velocity
L	length of plate	δ	location at which $U_x/U_\infty = 0.99$

Example 12.3. Water flows over a flat plate at a velocity U_∞ of 3 m s^{-1}. Calculate the total drag on a section of the plate that is 1 m wide and 2 m long, the beginning of the section coinciding with the leading edge of the plate. Assume the transition occurs at a Reynolds number of 5 x 16. Also, calculate the shear stress at the wall (in units of N m^{-2}) at a distance of 1 m from the leading edge of the plate.

Answer. The first step is to calculate $N_{Re,x}$ at distances of 1 m and 2 m from the leading edge. From Table A.l, the properties of water at room temperature (20°C or 293.15 K) are

$$\rho = \frac{1}{v_f} = \frac{1}{1.001 \times 10^{-3}} = 999 \text{ kg m}^{-3} \tag{i}$$

$$\mu = 1 \text{ cP} = 0.001 \text{ kg m}^{-1} \text{ s}^{-1}$$

At x = 1 m and x = 2 m, the Reynolds numbers from Eq. (6.3) are

$$N_{Re,x}(x = 1) = (1.0)(3.0)(999)/(0.001) = 2.997 \times 10^6$$
$$N_{Re,L}(x = 2) = (2.0)(3.0)(999)/(0.001) = 5.994 \times 10^6 \tag{ii}$$

The flow is clearly turbulent at both locations. The shear stress at the wall is given by Eq. (12.22):

$$\tau_w = \tfrac{1}{2}(\rho U_\infty^2)[2 \log_{10}(N_{Re,x}) - 0.65]^{-2.3}$$
$$= \tfrac{1}{2}(999)(3.0)^2[2 \log_{10}(2.997 \times 10^6) - 0.65]^{-2.3}$$
$$= 14.0 \,[(\text{kg m}^{-3})(\text{m}^2 \text{ s}^{-2})] = 14.0 \text{ kg m}^{-1} \text{ s}^{-2} = 14.0 \text{ N m}^{-2} \tag{iii}$$

The easiest approach is to calculate C_D from Eq. (12.24). For this problem B = 1700 from Table 12.1:

$$C_D = 0.455[\log_{10}(N_{Re,L})]^{-2.58} - B/(N_{Re,L})$$
$$= 0.455[\log_{10}(5.994 \times 10^6)]^{-2.58} - (1700)/(5.994 \times 10^6) = 0.00298 \tag{iv}$$

From Eq. (12.20), the drag on the plate is

$$F = \tfrac{1}{2}(\rho U_\infty^2)(Lb)(C_D) = \tfrac{1}{2}(999)(3.0)^2(2.0)(0.00298)$$
$$= 26.8 \,[(\text{kg m}^{-3})(\text{m}^2 \text{ s}^{-2})(\text{m}^2)] = 26.8 \text{ kg m s}^{-2} = 26.8 \text{ N} \tag{v}$$

The drag force on the plate can alternatively be calculated from Eq. (12.25):

$$F_{actual} = F_{laminar,0-x_c} + F_{turbulent,0-L} - F_{turbulent,0-x_c} \tag{12.25}$$

$F_{laminar,0-x_c}$ evaluated from Eq. (12.14) from $x = 0$ to $x = x_c$ (at $N_{Re,c}$)

$F_{turbulent,0-L}$ evaluated from Eqs. (12.20) and (12.21) from $x = 0$ to $x = L$

$F_{turbulent,0-x_c}$ evaluated from Eqs. (12.20) and (12.21) from $x = 0$ to $x = x_c$.

The first step is to locate the point of transition x_c, which can be found from the definition of $N_{Re,x}$, Eq. (6.3), where $N_{Re,c}$ is 5 X ld:

$$x_c = (N_{Re,c}\,\mu)/(U_\infty \rho) = [(5 \times 10^5)(0.001)]/[(3.0)(999)] = 0.167 \text{ m} \tag{vi}$$

In other words, the boundary layer is laminar for the first 16.7 cm, the point at which the transition to turbulence begins. Next, the laminar drag coefficient and the force on the plate are calculated from the leading edge ($x = 0$) to x_c (0.167 m). Equation (12.16), evaluated at x_c, yields

$$C_{D,\text{laminar}} = 4C_1(N_{Re,x})^{-1/2} = (4)(0.33206)(5 \times 10^5)^{-1/2} = 0.00188 \quad \text{(vii)}$$

This result is used in Eq. (12.15), which is solved for the drag force F:

$$F_{\text{laminar},0-x_c} = \tfrac{1}{2}(\rho U_\infty^2)(Lb)(C_{D,\text{laminar}})$$
$$= \tfrac{1}{2}(999)(3.0)^2(0.167)(1.0)(0.00188)\,[(\text{kg m}^{-3})(\text{m}^2\,\text{s}^{-2})(\text{m})(\text{m})]$$
$$= 1.411 \text{ kg m s}^{-2} = 1.411 \text{ N} \quad \text{(viii)}$$

The last two terms in Eq. (12.25) are calculated from Eqs. (12.20) and (12.21) using the lengths indicated in the subscripts. The middle term is

$$C_D = 0.455[\log_{10}(N_{Re,L})]^{-2.58}$$
$$= 0.455[\log_{10}(5.994 \times 10^6)]^{-2.58} = 0.003\,264 \quad \text{(ix)}$$
$$F_{\text{turbulent},0-L} = \tfrac{1}{2}(\rho U_\infty^2)(Lb)(C_D)$$
$$= \tfrac{1}{2}(999)(3.0)^2(2.0)(1.0)(0.003\,264) = 29.35 \text{ N} \quad \text{(x)}$$

For the length 0-x_c, the drag force is

$$C_D = 0.455[\log_{10}(N_{Re,c})]^{-2.58} = 0.455[\log_{10}(5 \times 10^5)]^{-2.58} = 0.005\,106 \quad \text{(xi)}$$
$$F_{\text{turbulent},0-x_c} = \tfrac{1}{2}(\rho U_\infty^2)(Lb)(C_D)$$
$$= \tfrac{1}{2}(999)(3.0)^2(0.167)(1.0)(0.005\,106) = 3.833 \text{ N} \quad \text{(xii)}$$

Substitution of the results from Eqs. (viii), (x), and (xii) into Eq. (12.25) yields the total drag on the plate:

$$F_{\text{actual}} = 1.411 + 29.35 - 3.833 = 26.93 \text{ N} \quad \text{(xiii)}$$

This value agrees well with that obtained by the short procedure, Eq. (12.24), i.e., 26.8 N in Eq. (v).

Example 12.4. Calculate the entry lengths in pipe flow for 99percent development of the velocity profile at turbulent Reynolds numbers of 2100, 4000, lo’, and 16. Compare these to the laminar length that might be obtained under very

TABLE U.3
Entry lengths for Example 12.4

Reynolds number $N_{Re,x}$	L_e/d_o Laminar $0.0567N_{Re}$	L_e/d_o Turbulent $0.693(N_{Re})^{1/4}$
2100	119	4.1
4000	227	5.5
10^4	567	6.9
16	5670	12.3

unusual conditions where no instabilities exist to trigger the transition to turbulence.

Answer. **At Reynolds numbers of 2100 and 4000, the flow in most equipment is really neither fully laminar nor fully turbulent. Therefore, the correct entry length is open to question. At Reynolds numbers of 10^4 and 10^5, the flow is almost always fully turbulent, and Eq. (12.26) applies.**

The results from Eq. (12.19) for laminar entry and from Eq. (12.26) for turbulent entry are compared in Table 12.3. The entry lengths for turbulent flow are quite small.

12.1.3 Heat and Mass Transfer During Boundary Layer Flow Past a Flat Plate

Of particular interest are problems of mass and heat transfer in boundary layer flows past a flat plate [El, S1]. The simplified momentum equations for steady-state laminar flow have already been given as Eqs. (12.1) and (12.3), which must now be coupled with an additional equation for either heat or mass transfer, or both. The appropriate simplifications of the general equations, Eq. (5.13) or Eq. (5.8), were detailed in Example 5.8. The results were

$$U_x \frac{\partial T}{\partial x} + U_y \frac{\partial T}{\partial y} = \alpha \frac{\partial^2 T}{\partial y^2} \tag{12.27}$$

$$U_x \frac{\partial C_A}{\partial x} + U_y \frac{\partial C_A}{\partial y} = D \frac{\partial^2 C_A}{\partial y^2} \tag{12.28}$$

These equations are similar to Eq. (12.1), the momentum equation:

$$U_x \frac{\partial U_x}{\partial x} + U_y \frac{\partial U_x}{\partial y} = \nu \frac{\partial^2 U_x}{\partial y^2} \tag{12.1}$$

Reviewing briefly, Figure 12.1 presented the solution to the coupled differential equations, Eq. (12.1) and Eq. (12.3), which were solved together through the introduction of a similarity transformation. If there is also heat transfer as well as momentum transfer, then Eqs. (12.1), (12.3), and (12.27) must all be solved together. If there is heat, mass, and momentum transfer all in the same problem, then Eqs. (12.1), (12.3), (12.27) and (12.28) form a "coupled set" and must be solved together.

Heat transfer. The simplest heat transfer problem is the case of fluid at U_∞ and T_∞ passing over a flat plate that is maintained at a constant and uniform temperature T_w. Even this problem is not solvable directly. The usual assumption is to "decouple" the partial differential equations by assuming that T_w differs only slightly from T_∞ so that the solution of Eqs. (12.1) and (12.3) in Fig. 12.1 remains valid. In actual practice, the presence of a temperature gradient alters the velocity profile.