PROCESS FLUID MECHANICS

By Morton M. Denn, *University of Delaware*

PROCESS FLUID MECHANICS provides a fully comprehensive, detailed, and orderly treatment of the essentials of fluid mechanics both from the macroscopic and microscopic viewpoints. Author Morton M. Denn has organized his excellent treatment into an introductory segment; four sections devoted to the subjects of dimensional analysis and experimentation, macroscopic balances, detailed flow structure, approximate methods, and a final section dealing with advanced topics.

Opening chapters discuss the nature of macro- and microscopic flow problems, physical units employed, and physical properties. This foundation is followed by the section on dimensional analysis and experimentation, which includes chapters on pipe flow and the flow of particulates, including flow through porous media. Part three presents clear discussions of macroscopic balances and their practical applications.

Detailed flow structure is treated in the chapters on microscopic balances, one-dimensional flows, accelerating flow, and converging flow. The fifth section of PROCESS FLUID MECHANICS addresses approximate methods, with coverage of ordering and approximation; creeping flow; the lubrication approximation; stream function, vorticity and potential flow; and the boundary layer approximation. The final section treats turbulence, perturbation and numerical solution, two-phase gas-liquid flow, and viscoelasticity.

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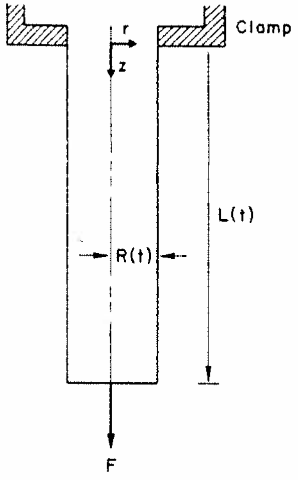
endent of r and θ, and obtain an expression for v_r in terms of v_z.
that p is independent of r and equals $-\eta\, dv_z/dz$. (Hint: Consider
at the outer radius of the liquid column.)
that $v_z = \Gamma z$, where Γ may be a function of t.
a constant imposed force, show that $L/L_0 = (1 - Ft/3A_0\eta)^{-1}$, where
and A_0 are the initial length and area of the liquid column.
analysis is a first step in describing the process of continuous filament
to form fibers.)



bubble is formed and grows in a large container of an isothermal liquid. The pressure in the bubble (relative to the quiescent from the bubble) can be related thermodynamically to the superheat for the bubble to grow. Obtain the relation between the bubble radius time, $R(t)$, and the bubble pressure. Inertia of the fluid being pushed by the growing bubble may be neglected.

commercial compression molding a cold fluid may be placed between hot causing a viscosity gradient. As a first approximation to this problem, the analysis for the squeeze film, Sec. 12.4, for the case in which the is a known function of position between the plates: $\eta = \eta(z)$. (v_r expressed explicitly in terms of an integral of a function of $\eta(z)$ for the which η is symmetric about the center plane. The problem is slightly if the origin is taken at the center plane, with each disk moving towards plane with velocity $V/2$.)

The Lubrication Approximation

13

13.1 INTRODUCTION

The lubrication approximation is a simplification that applies to flow between "nearly parallel" surfaces. This approximation was first used by Reynolds in 1886 in a study of lubrication, hence the name. The common name of the procedure is unfortunate, however, for it implies an unduly restrictive range of applications; the lubrication approximation is fundamental to the study of polymer processing, where it forms the basis for the analysis of extrusion, coating, calendering, and molding operations. It is, therefore, one of the most important methods of approximate solution of the Navier–Stokes equations.

We shall first introduce the lubrication approximation in an intuitive manner, and this introduction may suffice for some readers. We shall then develop the formalism through a more careful ordering analysis and consider further applications.

13.2 INTUITIVE DEVELOPMENT

Consider the pressure-driven flow of an incompressible Newtonian fluid between converging planes with a very small half-angle α, shown in Fig. 13-1. If the walls were perfectly parallel ($\alpha = 0$) this would simply be the problem of plane Poiseuille flow studied in Sec. 8.2, with a parabolic velocity profile

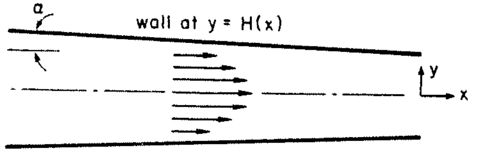


Figure 13-1. Schematic of pressure-driven flow in a plane channel with slowly converging walls.

given by Eq. (8.12):

$$v_x = \frac{H^2}{8\eta}\left(-\frac{d\mathcal{P}}{dx}\right)\left[1 - \left(\frac{2y}{H}\right)^2\right] \tag{13.1}$$

(We have retained $d\mathcal{P}/dx$ in place of $\Delta\mathcal{P}/L$ for reasons that will become obvious shortly.) Similarly, using Eq. (8.14) we may write the flow rate per unit width, q, as

$$q = \frac{H^3}{12\eta}\left(-\frac{d\mathcal{P}}{dx}\right) \tag{13.2}$$

If the angle α is small, we may reasonably expect the flow to be nearly the same as flow between parallel walls, so we expect Eqs. (13.1) and (13.2) to apply, *except that H is now a function of x*. The flow rate is a constant at all values of x, so it follows from Eq. (13.2) that $d\mathcal{P}/dx$ must vary as $H^{-3}(x)$ and cannot be a constant.

The power consumption depends on evaluation of σ_{xx}; compare Sec. 10.1. The viscous stress term $2\eta\, \partial v_x/\partial x$ is zero for flow between parallel walls, so the power consumption is simply equal to the product of flow rate and pressure drop. For flow between nearly parallel walls we may obtain $\partial v_x/\partial x$ by differentiation of Eq. (13.1); the result is proportional to dH/dx, which is equal to α for small α, and hence may be neglected. To within the small-angle approximation, therefore, the power consumption is again determined solely by the pressure drop. The pressure drop is obtained by rewriting Eq. (13.2) as an equation for $\mathcal{P}$,

$$\frac{d\mathcal{P}}{dx} = -\frac{12\eta q}{H^3(x)} \tag{13.3}$$

This is integrated to

$$\Delta\mathcal{P} = -12\eta q \int_{x_1}^{x_2} H^{-3}(x)\, dx \tag{13.4}$$

with a power consumption

$$P = q\,|\Delta\mathcal{P}| = 12\eta q^2 \int_{x_1}^{x_2} H^{-3}(x)\, dx \tag{13.5}$$

When the equation for the linear function $H(x)$ is substituted into Eq. (13.5), we simply obtain Eq. (10.66). This is plotted in Fig. 10-8, where we see that the assumption of nearly parallel flow is good up to a half-angle of about 15°.

It should be noted that nothing in the derivation of Eq. (13.5) in fact requires that $H(x)$ be linear. The conduit could have a nonlinear shape, as shown in Fig. 13-2, as long as the slope dH/dx is always small. In that case we can still compute $\Delta\mathcal{P}$ and P as long as the function $H(x)$ is known. We require separately, of course, that the length of the conduit be long compared to the average spacing between the walls, so that we may assume fully developed flow at all positions.

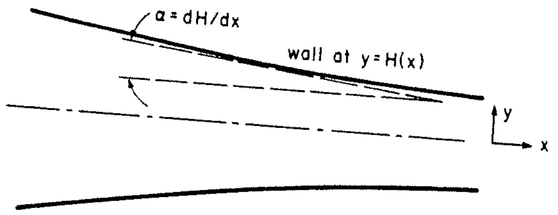


Figure 13-2. Schematic of a channel with slowly converging curved walls.

13.3 ORDERING ANALYSIS

We will develop the lubrication approximation for two-dimensional plane flows, but, with obvious changes appropriate to a different coordinate system, the results will clearly apply to axisymmetric three-dimensional flows as well. The geometry shown schematically in Fig. 13-3 is the basis for the ordering analysis. The lower surface moves at velocity U relative to the upper surface. The geometrical parameters satisfy the following inequalities:

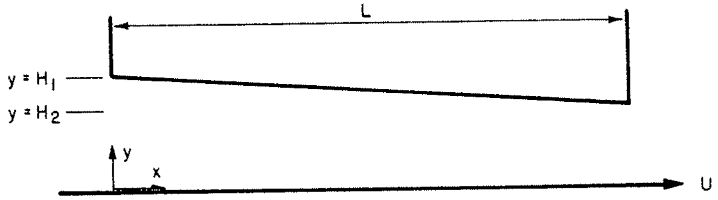


Figure 13-3. Schematic of a slider block. The bottom surface moves relative to the upper surface with velocity U. The spacing and change in spacing are both small relative to L.

$$\frac{H_1}{L} \ll 1 \tag{13.6a}$$

$$\frac{H_1 - H_2}{L} \ll 1 \tag{13.6b}$$

Equation (13.6a) is the assumption of a long flow channel, and Eq. (13.6b) is the "nearly parallel" assumption. We will be dealing at all times with flows for which the inertial terms are negligible,* as we shall show in Sec. 13.7. Our starting point is therefore the creeping flow equations for an incompressible Newtonian fluid, which for a two-dimensional planar flow have the form

$$\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} = 0 \tag{13.7}$$

$$0 = -\frac{\partial \mathcal{P}}{\partial x} + \eta\left(\frac{\partial^2 v_x}{\partial x^2} + \frac{\partial^2 v_x}{\partial y^2}\right) \tag{13.8a}$$

$$0 = -\frac{\partial \mathcal{P}}{\partial y} + \eta\left(\frac{\partial^2 v_y}{\partial x^2} + \frac{\partial^2 v_y}{\partial y^2}\right) \tag{13.8b}$$

To carry out an ordering analysis we will need to express the flow equations in dimensionless form. The significant feature of this problem is that it contains *two* characteristic lengths. The characteristic length in the y direction is clearly a spacing between the surfaces; to be specific we may take H_1, since H_1 and H_2 do not differ significantly. There are also changes in the x direction, however, and these take place over a distance of order L; thus, L is the characteristic length in the x direction. The dimensionless coordinates are therefore

$$\tilde{x} = \frac{x}{L} \qquad \tilde{y} = \frac{y}{H_1} \tag{13.9}$$

The characteristic velocity in the flow direction is clearly the linear velocity of the lower surface, U. There will also be some flow in the y direction, however, since the walls are not parallel. This y-direction flow will be characterized by a velocity that is different from U, and we shall denote it as V. V is an unknown at this stage of the development and must still be determined. Recognizing this fact, we write the dimensionless velocity components as

$$\tilde{v}_x = \frac{v_x}{U} \qquad \tilde{v}_y = \frac{v_y}{V} \tag{13.10}$$

Finally, the dimensionless equivalent pressure is written

$$\tilde{\mathcal{P}} = \frac{\mathcal{P}}{\Pi} \tag{13.11}$$

The characteristic pressure, Π, must also be determined.

It is convenient to consider the dimensionless equations one at a time. The dimensionless continuity equation, Eq. (13.7), is

$$\frac{U}{L}\frac{\partial \tilde{v}_x}{\partial \tilde{x}} + \frac{V}{H_1}\frac{\partial \tilde{v}_y}{\partial \tilde{y}} = 0 \tag{13.12a}$$

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*Recall that the inertial terms vanish identically in perfectly parallel flows of the type studied in Chapter 8.

or

$$\left(\frac{UH_1}{VL}\right)\frac{\partial \tilde{v}_x}{\partial \tilde{x}} + \frac{\partial \tilde{v}_y}{\partial \tilde{y}} = 0 \tag{13.12b}$$

The dimensionless group UH_1/VL must be of order unity. This follows by considering the consequences of any other choice. If UH_1/VL is large compared to unity, then the $\partial\tilde{v}_x/\partial\tilde{x}$ term in the dimensionless equation (13.12) will dominate and $\partial\tilde{v}_y/\partial\tilde{y}$ can be neglected; in that case, the equation simplifies to $\partial\tilde{v}_x/\partial\tilde{x} = 0$, which contradicts the necessity of allowing v_x to vary with x as the spacing changes. Similarly, if UH_1/VL is very small compared to unity, we may neglect the $\partial\tilde{v}_x/\partial\tilde{x}$ term relative to $\partial\tilde{v}_y/\partial\tilde{y}$; in that case, the equation simplifies to $\partial\tilde{v}_y/\partial\tilde{y} = 0$, and the boundary condition requiring $\tilde{v}_y$ to vanish at the wall then requires that $\tilde{v}_y$ be zero everywhere, which is a contradiction in a changing cross section. The continuity equation therefore *defines* V by the requirement that UH_1/VL be of order unity:

$$V = \frac{UH_1}{L} \tag{13.13}$$

Note that, consistent with one's intuition, $V \ll U$ for this nearly one-dimensional flow.

We now turn to Eq. (13.8a), the x component of the momentum equation. In dimensionless form this is written

$$0 = -\frac{\Pi}{L}\frac{\partial \tilde{\mathcal{P}}}{\partial \tilde{x}} + \frac{\eta U}{H_1^2}\left[\left(\frac{H_1}{L}\right)^2 \frac{\partial^2 \tilde{v}_x}{\partial \tilde{x}^2} + \frac{\partial^2 \tilde{v}_x}{\partial \tilde{y}^2}\right] \tag{13.14}$$

One simplification is immediately obvious. Since $H_1/L \ll 1$, the x-derivative term in the brackets may be neglected relative to the y-derivative term. This is consistent with our intuitive understanding that rates of change in the y direction are much larger than rates of change in the x (flow) direction. We may thus rewrite Eq. (13.14) as

$$\left(\frac{\Pi H_1^2}{\eta UL}\right)\frac{\partial \tilde{\mathcal{P}}}{\partial \tilde{x}} = \frac{\partial^2 \tilde{v}_x}{\partial \tilde{y}^2} \tag{13.15}$$

The two terms in Eq. (13.15) must be of comparable magnitude, since neither term can dominate without introducing a contradiction; indeed, for parallel walls Eq. (13.15) is simply a statement of the balance between the pressure drop and shear stress terms. Thus, the dimensionless group $\Pi H_1^2/\eta UL$ must be of order unity, and we obtain an expression for the characteristic pressure:

$$\Pi = \frac{\eta UL}{H_1^2} \tag{13.16}$$

Finally, we consider the y component of the momentum equation, Eq. (13.8b). In dimensionless form, using Eq. (13.16) for Π, this is written

$$0 = -\frac{\eta UL}{H_1^3}\frac{\partial \tilde{\mathcal{P}}}{\partial \tilde{y}} + \frac{\eta U}{LH_1}\left[\left(\frac{H_1}{L}\right)^2 \frac{\partial^2 \tilde{v}_y}{\partial \tilde{x}^2} + \frac{\partial^2 \tilde{v}_y}{\partial \tilde{y}^2}\right] \tag{13.17}$$

As before, we may neglect the x-derivative term in the brackets relative to the y-derivative term. Thus, we can write Eq. (13.17) after some simplification as

$$\frac{\partial \tilde{\mathcal{P}}}{\partial \tilde{y}} = \left(\frac{H_1}{L}\right)^2 \frac{\partial^2 \tilde{v}_y}{\partial \tilde{y}^2} \simeq 0 \tag{13.18}$$

All the characteristic quantities have been defined, so there are no more degrees of freedom. We must therefore conclude from Eq. (13.18) that, to within the approximation that $H_1/L \ll 1$, *$\partial\tilde{\mathcal{P}}/\partial\tilde{y}$ is negligible and $\tilde{\mathcal{P}}$ is a function only of $\tilde{x}$*. This is a primary result—that we may neglect variations in the pressure over the width of the channel.

We can summarize the ordering analysis by rewriting Eqs. (13.15) and (13.18) in dimensional form:

$$\mathcal{P} = \mathcal{P}(x) \tag{13.19}$$

$$\frac{d\mathcal{P}}{dx} = \eta \frac{\partial^2 v_x}{\partial y^2} \tag{13.20}$$

These are the equations that describe flow between parallel walls, except that $d\mathcal{P}/dx$ need not be a constant and v_x may depend on x as well as on y.

13.4 LUBRICATION EQUATIONS

Equations (13.19) and (13.20) are the basic equations for the lubrication approximation. Because of their very simple structure they can be solved directly and expressed in alternative, more useful forms. Because $d\mathcal{P}/dx$ is independent of y, Eq. (13.20) can be integrated twice to give

$$v_x = \frac{1}{2\eta}\frac{d\mathcal{P}}{dx} y^2 + C_1 y + C_2 \tag{13.21}$$

The "constants" of integration C_1 and C_2 are independent of y, but they will depend on x. If we take the origin of the y coordinate at the moving surface, we have boundary conditions

$$\text{at } y = 0\text{:} \quad v_x = U \tag{13.22a}$$

$$\text{at } y = H(x)\text{:} \quad v_x = 0 \tag{13.22b}$$

C_1 and C_2 can then be evaluated to give

$$v_x = U\left[1 - \frac{y}{H(x)}\right] - \frac{1}{2\eta}\frac{d\mathcal{P}}{dx} yH(x)\left[1 - \frac{y}{H(x)}\right] \tag{13.23}$$

Equation (13.23) is simply the equation describing the velocity distribution between two flat plates with an imposed pressure gradient. In this case, however, *we do not know the pressure gradient*. This is a typical situation in applications of the lubrication approximation. We may not know the flow rate between the plates either, but we do know that it must be the same at all values of x. Defining q as the flow rate per unit width, we have

$$q = \int_0^{H(x)} v_x\, dy = \text{constant} \tag{13.24}$$

and, carrying out the integration of Eq. (13.23),

$$q = \frac{UH(x)}{2} - \frac{H^3(x)}{12\eta}\frac{d\mathcal{P}}{dx} \tag{13.25}$$

Equation (13.25) is sometimes taken as the starting point for the lubrication approximation. It can be looked upon as an equation for $d\mathcal{P}/dx$ and rearranged to

$$\frac{d\mathcal{P}}{dx} = 12\eta\left[\frac{U}{2H^2(x)} - \frac{q}{H^3(x)}\right] \tag{13.26}$$

or, integrating once,

$$\mathcal{P}(x) = \mathcal{P}_0 + 6\eta U \int_0^x \frac{dx}{H^2(x)} - 12\eta q \int_0^x \frac{dx}{H^3(x)} \tag{13.27}$$

$\mathcal{P}_0$ is a constant of integration that represents the pressure at $x = 0$. Finally, a useful expression relating the flow rate, q; the relative velocity, U; and the overall pressure change, $\mathcal{P}_0 - \mathcal{P}(L)$; can be obtained by setting $x = L$ in Eq. (13.27) and rearranging:

$$q = \frac{\mathcal{P}_0 - \mathcal{P}(L)}{12\eta \int_0^L H^{-3}(x)\, dx} + \frac{U}{2}\frac{\int_0^L H^{-2}(x)\, dx}{\int_0^L H^{-3}(x)\, dx} \tag{13.28}$$

For the case in which $U = 0$, this is simply Eq. (13.4). Note that although Fig. 13-3 is drawn with two plane surfaces, $H(x)$ can in fact be any function of x as long as dH/dx is small compared to unity.

13.5 COATING

We analyzed the problem of wire coating in Sec. 8.5. The treatment there was somewhat oversimplified in that we took the die to be of uniform cross section and assumed that the reservoir pressure was atmospheric. The real situation is more likely to be like that shown in Fig. 13-4, with a possible pressure drop between the reservoir and the die exit. We will analyze this flow here for the coating of a sheet rather than a wire, in keeping with the two-dimensional equations developed in this chapter. The case of a wire die is identical, except that the equations for axisymmetric flow are used. The sheet case applies to the wire as well when the maximum spacing between the wire and the die wall is small compared to the radius of the wire.

The coating thickness H_t is related to the flow rate by the equation

$$q = UH_t \tag{13.29}$$

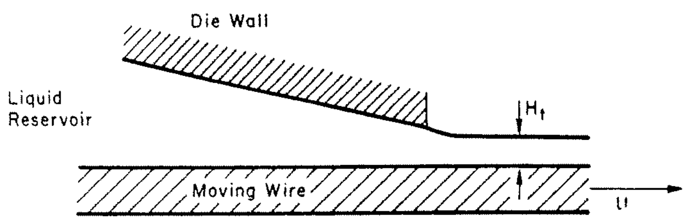

Figure 13-4. Schematic of a wire coating die.

We can therefore write Eq. (13.28) as

$$H_t = \frac{\mathcal{P}_0 - \mathcal{P}(L)}{12\eta U \int_0^L H^{-3}(x)\,dx} + \frac{1}{2}\frac{\int_0^L H^{-2}(x)\,dx}{\int_0^L H^{-3}(x)\,dx} \tag{13.30}$$

If there is no net pressure drop through the die $[\mathcal{P}_0 = \mathcal{P}(L)]$, the coating thickness depends only on die geometry. In general, however, a pressure drop across the die will be employed, and for a given sheet or wire speed the pressure drop determines the thickness. The pressure drop required for a given coating thickness is obtained by rewriting Eq. (13.30) in the form

$$\mathcal{P}_0 - \mathcal{P}(L) = 12\eta U \int_0^L \frac{1}{H^2(x)}\left[\frac{H_t}{H(x)} - \frac{1}{2}\right]dx \tag{13.31}$$

It is evident from Eq. (13.31) that the coating thickness can never be less than one-half the exit spacing of a converging die, or else a negative pressure drop would be required.

It is of interest to examine the velocity profile in the die. If we replace q in Eq. (13.26) with UH_t and substitute for $d\mathcal{P}/dx$ in Eq. (13.23), we can express the velocity at any position as

$$v_x = U\left(1 - \frac{y}{H}\right)\left[1 - 3\left(1 - \frac{2H_t}{H}\right)\frac{y}{H}\right] \tag{13.32}$$

It is understood in Eq. (13.32) that H is a function of x. The term in brackets will be negative over a portion of the cross section whenever $H > 3H_t$, indicating a negative velocity and a region of backflow near the wall, as shown schematically in Fig. 13-5. The region of zero net flow (flow forward exactly compensated by reverse flow) occurs in the region $H_0 \leq y \leq H$, where H_0 is defined by the equation

$$0 = \int_{H_0}^{H} v_x\,dy \tag{13.33}$$

When Eq. (13.32) is substituted into Eq. (13.33) and the integration carried out, H_0 is found to satisfy the equation

$$H_0(x) = \frac{H(x)H_t}{H(x) - 2H_t} \tag{13.34}$$

Equation (13.34) is plotted in Fig. 13-5 for a plane wall. Fluid in the region $y \leq H_0(x)$ is swept out and forms the coating, while fluid in the region $H_0(x) \leq y \leq H(x)$ simply recirculates. The recirculation will result in a long residence time in the die for a portion of the fluid and could lead to degradation of the coating material in some cases. Recirculation can be avoided by ensuring that the spacing between the moving surface and the die wall never exceeds $3H_t$.

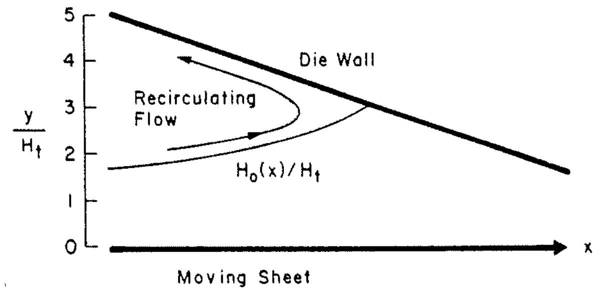

Figure 13-5. Recirculation in a coating die with a plane wall.

13.6 SLIDER BLOCK

The application of the lubrication approximation to a classical problem in lubrication is illustrated by reference to Fig. 13-3. We suppose that the stationary block is completely immersed in the fluid. The pressure in the fluid outside the space between the moving sheet and the block, but near $x = 0$ and $x = L$, is then simply hydrostatic. Thus, $\mathcal{P}(L) = \mathcal{P}_0$, and we may take $\mathcal{P}_0 = 0$ without any loss of generality. It then follows from substitution of Eq. (13.28) into (13.27) that

$$\mathcal{P}(x) = 6\eta U \int_0^x \frac{H(x) - \mathcal{H}}{H^3(x)}\,dx \tag{13.35}$$

where

$$\mathcal{H} = \frac{\int_0^L H^{-2}(x)\,dx}{\int_0^L H^{-3}(x)\,dx} \tag{13.36}$$

For the special case of a plane surface (a *wedge*),

$$\text{wedge:}\quad \mathcal{P}(x) = \frac{6\eta U}{H_1^2 - H_2^2}\,\frac{[H_1 - H(x)][H(x) - H_2]}{H^2(x)} \tag{13.37}$$

The upward stress σ_{yy} consists of a pressure term and a term $2\eta\,\partial v_y/\partial y$. From the continuity equation, $\partial v_y/\partial y$ is equal to $-\partial v_x/\partial x$, and the latter is proportional to dH/dx and can thus be neglected. The force per unit width acting upward on the block is therefore

$$F_N = \int_0^L \mathcal{P}(x)\, dx \tag{13.38}$$

For a wedge this force is

$$\text{wedge:}\quad F_N = \frac{6\eta UL^2}{H_1^2 - H_2^2}\left[\ln\left(\frac{H_1}{H_2}\right) - \frac{2(H_1 - H_2)}{H_1 + H_2}\right] \tag{13.39}$$

The normal force is nonzero only for $H_1 \neq H_2$. The force F_N takes on a maximum for $H_1 \simeq 2.2H_2$; in that case

$$\text{wedge:}\quad F_{N,\max} \simeq 0.16\, \frac{\eta UL^2}{H_2^2} \tag{13.40}$$

Clearly, a very large normal force is exerted by the thin film of liquid for small H_2. This is the basis of effective lubrication.

The force required to pull the plate past the block is obtained by integrating the shear stress at the plate:

$$F_s = -\int_0^L \eta \left.\frac{\partial v_x}{\partial y}\right|_{y=0} dx \tag{13.41}$$

(The negative sign is required because $\eta \partial v_x/\partial y$ is the stress exerted by the fluid on the plate, while we require the equal and opposite value.) For the special case of a wedge the integration gives

$$\text{wedge:}\quad F_s = \frac{\eta UL}{H_1 - H_2}\left[4 \ln \frac{H_1}{H_2} - \frac{6(H_1 - H_2)}{H_1 + H_2}\right] \tag{13.42}$$

At the ratio $H_1/H_2 = 2.2$, corresponding to the maximum upward force, we have

$$\text{wedge:}\quad F_{s,\max} \simeq 0.75\, \frac{\eta UL}{H^2} \tag{13.43}$$

We can then compute the *coefficient of friction*, the ratio of imposed shear force to obtained normal force, as

$$\text{coefficient of friction} = \frac{F_{s,\max}}{F_{N,\max}} \simeq 5\, \frac{H_2}{L} \tag{13.44}$$

This can be made a very small value for a sufficiently thin liquid film.

13.7 NEGLECT OF INERTIA

Our starting point in the derivation of the lubrication equations was taken to be the creeping flow equations. It is helpful to examine the inertial terms to ensure that they are indeed negligible.

The x component of the steady-state Navier–Stokes equations is

$$\rho\left(v_x \frac{\partial v_x}{\partial x} + v_y \frac{\partial v_x}{\partial y}\right) = -\frac{\partial \mathcal{P}}{\partial x} + \eta\left(\frac{\partial^2 v_x}{\partial x^2} + \frac{\partial^2 v_x}{\partial y^2}\right) \tag{13.45}$$

In dimensionless form, using V and Π as defined by Eqs. (13.13) and (13.16), respectively, we obtain

$$\frac{\rho U^2}{L}\left(\tilde{v}_x \frac{\partial \tilde{v}_x}{\partial \tilde{x}} + \tilde{v}_y \frac{\partial \tilde{v}_x}{\partial \tilde{y}}\right) = \frac{\eta UL}{H^2}\left[-\frac{\partial \tilde{\mathcal{P}}}{\partial \tilde{x}} + \left(\frac{H}{L}\right)^2 \frac{\partial^2 \tilde{v}_x}{\partial \tilde{x}^2} + \frac{\partial^2 \tilde{v}_x}{\partial \tilde{y}^2}\right] \tag{13.46}$$

Thus, the inertial terms will be negligible if

$$\frac{\rho U^2}{L} \ll \frac{\eta UL}{H^2} \qquad \frac{\rho U^2 \cdot H^2}{\eta L^2} \ll 1 \tag{13.47a}$$

or, equivalently,

$$\left(\frac{\rho UH}{\eta}\right)\frac{H}{L} = \mathrm{Re}\, \frac{H}{L} \ll 1 \tag{13.47b}$$

This includes a much wider range of flow conditions than the stronger requirement for general creeping flow, Re $\ll$ 1.

13.8 CONCLUDING REMARKS

The lubrication approximation illustrates the way in which order-of-magnitude estimates can be used to simplify flow problems when the flow is "almost" one-dimensional, providing two characteristic length scales and hence two characteristic velocities. The same type of ordering will be used again in Chapter 15 on boundary layer flows. The use of the lubrication approximation has led to considerable insight into both lubrication flows and polymer processing operations. Successful application to polymer processing has been achieved because processing applications such as extrusion, molding, coating, and calendering involve the flow of a very viscous liquid between surfaces that are in close proximity and in relative motion with respect to one another.

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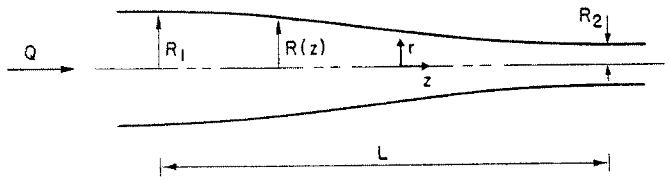
For applications in polymer processing, see

McKelvey, J. M., *Polymer Processing*, John Wiley & Sons, Inc., New York, 1962.
Middleman, S., *Fundamentals of Polymer Processing*, McGraw-Hill Book Company, New York, 1977.
Pearson, J. R. A., *Mechanical Principles of Polymer Melt Processing*, Pergamon Press, Oxford, 1965.

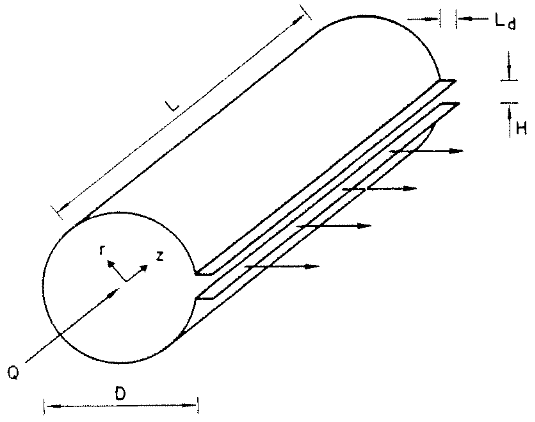
PROBLEMS

13.1. Estimate the pressure drop in the slowly-varying cylindrical contraction shown in Fig. 13P1.



13.2. Repeat Problems 12.3 and 12.4 for a case in which the spacing H between the disks is a slowly varying function of radius, $H(r)$.

13.3. An end-fed sheeting die is shown schematically in Fig. 13P3. Fluid flows axially because of an axial pressure gradient. There is also a side flow through the die because of the pressure difference between the tube and the die exit. Estimate the flow distribution along the length of the die. You may assume that the axial tube flow is locally fully-developed Poiseuille flow, and that the die flow rate at each position is fully-developed plane Poiseuille flow. (A nearly identical analysis applies to slow leakage through a porous-walled tube, as in flow in the kidney.)



Stream Function, Vorticity, and Potential Flow

14

14.1 INTRODUCTION

Three auxiliary functions, the *stream function*, the *vorticity*, and the *potential function*, are often used to represent and interpret solutions of fluid mechanics problems. The stream function is applicable to certain incompressible flows. Vorticity is a measure of local rotation and is broadly used. The potential function is important in the inviscid limit. We shall briefly introduce these concepts in this chapter.

14.2 STREAM FUNCTION

Consider a two-dimensional plane flow of an incompressible fluid with $v_z = 0$. The continuity equation is

$$\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} = 0 \qquad (14.1)$$

The *stream function* $\psi(x, y)$ is defined as the function such that

$$v_x = -\frac{\partial \psi}{\partial y} \qquad v_y = +\frac{\partial \psi}{\partial x} \qquad (14.2)$$

Clearly, the continuity equation is automatically satisfied when Eqs. (14.2) are substituted into Eq. (14.1). The equivalent relations in cylindrical coor-