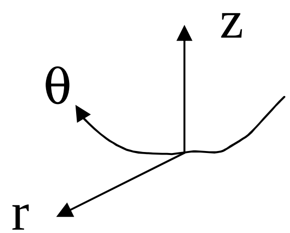
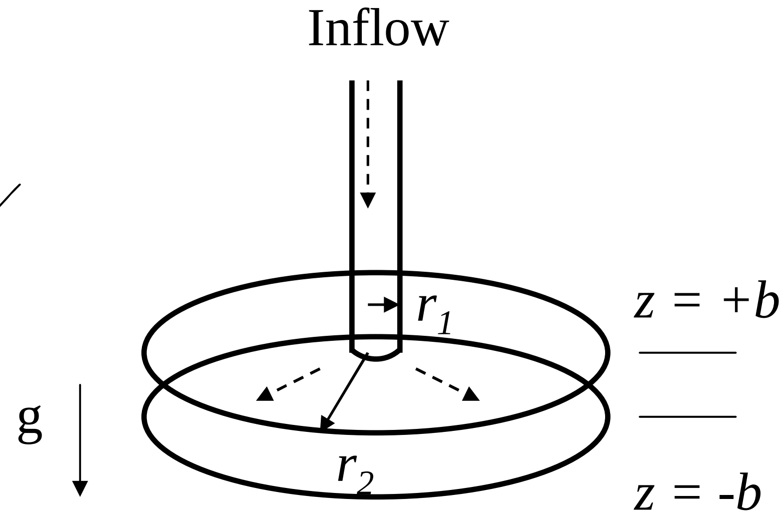
Radial Flow Between Two Parallel Discs

- select appropriate co-ordinate system (cylindrical polars)
- very viscous flow
- slow flow
- incompressible
- steady flow (no time derivative)
- within gap $\frac{\partial}{\partial \theta} = 0, u_\theta = 0, u_z = 0$





Continuity

$$\frac{1}{r}\frac{\partial (r u_r)}{\partial r} + \underbrace{\frac{1}{r}\frac{\partial u_\theta}{\partial \theta}}_{=0} + \underbrace{\frac{\partial u_z}{\partial z}}_{=0} = 0 \Rightarrow \frac{1}{r}\frac{\partial (r u_r)}{\partial r} = 0 \Rightarrow r \cdot u_r = \varphi(z) \neq f(r, \theta)$$

z-component

$$\rho\left(\underbrace{\frac{\partial u_z}{\partial t}}_{=0} + \underbrace{u_r \frac{\partial u_z}{\partial r}}_{=0} + \underbrace{\frac{u_\theta}{r}\frac{\partial u_z}{\partial \theta}}_{=0} + u_z \frac{\partial u_z}{\partial z}\right) = -\frac{\partial P}{\partial z} + \mu\left[\underbrace{\frac{1}{r}\frac{\partial}{\partial r}\left(r \frac{\partial u_z}{\partial r}\right)}_{=0} + \underbrace{\frac{1}{r^2}\frac{\partial^2 u_z}{\partial \theta^2}}_{=0} + \underbrace{\frac{\partial^2 u_z}{\partial z^2}}_{=0}\right] + \rho g_z$$

= 0 steady flow

$$\Rightarrow \frac{\partial P}{\partial z} = -\rho g$$

r-component

= 0 steady flow   = 0   = 0 by continuity   = 0

$$\rho\left(\frac{\partial u_r}{\partial t}+u_r\frac{\partial u_r}{\partial r}+\frac{u_\theta}{r}\frac{\partial u_r}{\partial \theta}-\frac{u_\theta^2}{r}+u_z\frac{\partial u_r}{\partial z}\right)=-\frac{\partial P}{\partial r}+\mu\left[\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial (r u_r)}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 u_r}{\partial \theta^2}-\frac{2}{r^2}\frac{\partial u_\theta}{\partial \theta}+\frac{\partial^2 u_r}{\partial z^2}\right]+\rho g_r$$

$u_\theta = 0$ $\frac{\partial}{\partial \theta}=0$

$$\Rightarrow \rho u_r\frac{\partial u_r}{\partial r}=-\frac{\partial P}{\partial r}+\mu\frac{\partial^2 u_r}{\partial z^2}$$

Now $r \cdot u_r = \phi(z)$ $\Rightarrow \frac{\partial u_r}{\partial r}=-\frac{\phi}{r^2}\Rightarrow u_r\frac{\partial u_r}{\partial r}=-\frac{\phi^2}{r^3}$

$$\Rightarrow -\rho\frac{\phi^2}{r^3}=-\frac{\partial P}{\partial r}+\frac{\mu}{r}\frac{\partial^2\phi}{\partial z^2}$$

If we have creeping flow, Re<<1, giving:

$$\frac{\partial P}{\partial r}=\frac{\mu}{r}\frac{\partial^2\phi}{\partial z^2}$$

Integrate with respect to r between r_1 and r_2

Pressure inside pipe (in excess of external pressure)

$$\mu \frac{\partial^2 \phi}{\partial z^2} \int_{r_1}^{r_2} \frac{dr}{r} = \int_0^{-\Delta P} dP \Rightarrow \Delta P = -\mu \ln\left(\frac{r_2}{r_1}\right)\frac{\partial^2 \phi}{\partial z^2}$$

Integrate with respect to z twice

$$\Delta P\left(\frac{z^2}{2} + Az + B\right) = -\mu \ln\left(\frac{r_2}{r_1}\right)\phi$$

Boundary conditions: at $z = 0$, by symmetry,

$$\frac{\partial u_r}{\partial z} = 0 \Rightarrow A = 0$$

at $z = +/-b$, no slip condition, $u_r = 0$,

$$\Rightarrow B = -\frac{b^2}{2}$$

$$\Rightarrow u(r,z) = \frac{\Delta P \cdot b^2}{2\mu \cdot r \cdot \ln\left(r_2/r_1\right)} \cdot \left[1 - \left(\frac{z}{b}\right)^2\right]$$